Homoclinics and chaotic behaviour for perturbed second order systems

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Abstract

This paper deals with perturbed dynamical systems of the form:

$$-\ddot{u} + u = \nabla V(u) + \epsilon \nabla_u W(t, u)$$

where $u(t) \in \mathbb{R}^n (n \geq 1)$. By means of a variational approach the existence of multibump homoclinics is proved under general assumptions on the Melnikov function. As a particular case, if $W(\cdot, u)$ is T-periodic, the existence of approximate and complete Bernoulli shift structures is proved. An application to partial differential equations is also given. [1]

1 Introduction

This paper deals with homoclinics and chaotic behaviour for perturbed dynamical systems and partial differential equations.

From the works of Poincaré [14] it became clear that the existence of homoclinic orbits determines a chaotic behaviour in the dynamics of a system. Consider a symplectic diffeomorphism $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with a hyperbolic fixed point p. The intersection points between the stable and the unstable manifolds W^s and W^u are called homoclinic points. Poincaré proved in [14] that if W^s and W^u intersect transversally then the diffeomorphism Φ admits infinitely many homoclinic points. This result was later improved by Birkhoff and Smale. They proved that in presence of a transverse homoclinic point $r \neq p$, the map Φ admits a Bernoulli shift structure. In particular it implies sensitive dependence on initial conditions and more precisely that the topological entropy of Φ is positive.

The Smale-Birkhoff theorem can be applied to T-periodic Hamiltonian systems through the Poincaré map defined as the time-T map of the Hamiltonian vector field.

For small perturbations of autonomous Hamiltonian systems the transversality condition can be checked using the Melnikov function. The existence of simple zeros of the this function implies the existence of transverse intersections.

All the former results are obtained by analytical methods.

In recent years with the works of Bolotin [7] and Coti-Zelati-Ekeland-Séré [8] variational methods too have been successfully applied for the search of homoclinics. In [16] E. Séré developed new variational technics to prove, under global assumptions, the existence of infinitely many homoclinic solutions of "multibump" type. Generalizing [16] the same author proved in [17] the existence of an approximate Bernoulli shift structure; this is sufficient to show that the topological entropy of the system is positive and hence that the dynamics of the system is chaotic. Other papers extend

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the existence and multiplicity results of [8],[10],[16] and [17] to almost-periodic systems (see for example [9],[13] and [15]).

Variational methods have been applied in the perturbative case too.

For example in [6] Bessi constructed, for the 1-dimensional pendulum with a periodic forcing term, an approximate Bernoulli shift structure under the assumption that the Melnikov function is non-constant.

More recently Ambrosetti and Badiale [1] used the Melnikov function to prove existence and multiplicity results for homoclinics in perturbed Lagrangian systems and partial differential equations of Schrödinger type.

In order to have an idea of the method used in [1] let us consider second order Lagrangian systems with n degrees of freedom of the form:

$$-\ddot{u} + u = \nabla V(u) + \epsilon \nabla_u W(t,u) \tag{1.1}$$

where $V(0) = 0$, $\nabla V(0) = 0$, $D^2V(0) = 0$ and $W(t,0) = 0$, $\nabla_u W(t,0) = 0$. Homoclinic solutions of (1.1) are critical points of the Lagrangian functional:

$$f_\epsilon(u) = \int_{\mathbb{R}} \frac{|u|^2}{2} + \frac{|\dot{u}|^2}{2} - V(u) - \epsilon W(t,u)dt.$$

If the unperturbed equation has a homoclinic $u_0 \neq 0$ then the set $Z = \{u_\theta = u_0(\cdot + \theta)\}$ is a manifold of critical points for f_0.

Ambrosetti and Badiale in [1] look for the existence of critical points of f_ϵ near the 1-dimensional manifold Z. They show that critical points of f_ϵ close to Z can be found as critical points of the restriction $f_{\epsilon|Z_\epsilon}$ where Z_ϵ is a 1-dimensional manifold close to Z (Z_ϵ is called a natural constraint). It turns out that $f_{\epsilon|Z_\epsilon}$ is, up to a constant, very close to the function $\epsilon\Gamma$ where Γ is defined by:

$$\Gamma(\theta) = -\int_{\mathbb{R}} W(t, u_0(t+\theta))dt$$

This is nothing but the primitive of the Melnikov function and it will be called throughout this paper the Melnikov primitive.

It follows that, roughly speaking, critical points of the Melnikov primitive Γ give rise to critical points of the functional f_ϵ.

In this paper we generalize their method in order to prove the existence of multibump homoclinic solutions and chaotic behaviour in such systems. We prove a connection between multiplicity results for homoclinics and the properties of the Melnikov primitive. Roughly speaking when the Melnikov primitive has critical points which are sufficiently separated there exist multibump solutions which can be located. We prove this result showing that it is possible to construct, for ϵ small enough, k-dimensional constrained manifolds, such that the critical points of the restriction of f_ϵ to these manifolds give rise to k-bump homoclinic solutions of (1.1). Even if in general the unperturbed functional f_0 does not possess a k-dimensional manifold of critical points such a construction can be performed because:

$${f_0}'(u(\theta_1 + \cdot) + \ldots + u(\theta_k + \cdot)) \to 0$$

as $\min_i(\theta_{i+1} - \theta_i) \to +\infty$.

Still in this situation, particular properties of the Melnikov primitive Γ induce the existence of homoclinics u_ϵ; each bump of u_ϵ is located near $u_0(\theta + \cdot)$ for some critical point θ of Γ.

Moreover, using in the computations estimates which do not depend on the number of bumps k, we obtain the existence of solutions of (1.1) with infinitely many bumps. See theorems 4 and 5, which are the main results of this paper.

When the perturbation is periodic, a dynamical consequence of the existence of these multibump solutions is the presence of an approximate Bernoulli shift structure (see [17]); in addition, if the Melnikov primitive possesses at least one non-degenerate critical point then we prove by means of this variational method a classical result, namely the existence of a complete (continuous) Bernoulli shift structure in the dynamics of the system.

We underline that an advantage of the method is that, since all our results are obtained in the same abstract variational setting, we do not require any general restriction on the time-dependence of the perturbation such as periodicity, almost-periodicity, etc. If the perturbation is almost periodic the condition that the Melnikov primitive is non constant is sufficient to guarantee the existence of infinitely many homoclinics and of solutions with infinitely many bumps.

Moreover our method is well suited also to study partial differential equations of Schrödinger type.

For the sake of clarity we prefer to prove, first, the existence of 2-bump solutions, developing all the computations in a case in which the technicalities are as small as possible. We underline that the existence of k-bump solutions and of solutions with infinitely many bumps do not require any stronger assumption on the Melnikov primitive.

The paper is organized as follows:

Section 2 is devoted to the proof, whenever the Melnikov primitive satisfies suitable conditions, of the existence of infinitely many 2-bump solutions.

Section 3 deals with solutions with infinitely many bumps. In addition a uniqueness result is proved under the assumption that the Melnikov primitive possesses non-degenerate critical points.

In Section 4 we study some consequences of the results of section 3 in the periodic case, such as the existence of a Bernoulli shift structure.

Section 5 concerns other second-order systems and an application of the method to partial differential equations of Schrödinger type.

Contents

Notations

- $\nabla V = (D_i V)_{1\le i\le n}$, the gradient of V;
- D^k, the k-th derivative;
- f', the gradient of the functional f;
- f'', the second derivative of the functional f;
- $E = W^{1,2}(\mathbb{R}^m, \mathbb{R}^n)$, the standard Sobolev space with the scalar product $(\cdot,\cdot)$ defined by $(u,v) = \int_{\mathbb{R}^m} (\nabla u \nabla v + uv)$ and the norm $||u||^2 = (u,u)$.
- We shall use the continuous embeddings $E \hookrightarrow L^q(\mathbb{R}^m, \mathbb{R}^n)$ where $2 \le q \le 2^* = \frac{2m}{m-2}$ if $n \ge 3$, $2 \le q < \infty$ if $m = 2$ and $2 \le q \le \infty$ if $m = 1$;
- $u_n \rightharpoonup u$ and $u_n \to u$ will mean that u_n converges respectively in the weak and in the strong topology to u in E;

- There is an action of $\mathbb{R}^m$ on E defined by $\theta * u = u(\theta + \cdot)$ which preserves the scalar product $(\cdot, \cdot)$;
- $\langle \phi_1, \dots, \phi_m \rangle := span\{\phi_1, \dots, \phi_m\} = \{\alpha_1 \phi_1 + \cdots + \alpha_m \phi_m \mid \alpha_i \in \mathbb{R}\}$.

C, C', C'' will denote in the proofs positive constants which do not depend of anything, but which can take each time a different value.

The notation C_i will be reserved to positive constants which appear in the lemmas and which have a fixed value.

Moreover $o_L(1)$ (resp. $o_{L,\epsilon}(1)$) will denote a quantity which tends to 0 as $L \to +\infty$ (resp. as $L \to +\infty$ and $\epsilon \to 0$) independently of anything else.

The expression "$a(z_1, \dots, z_p) = O(b(z_1, \dots, z_p))$" will mean that there is an absolute positive constant C such that for all $(z_1, \dots, z_p)$, $|a(z_1, \dots, z_p)| \leq C|b(z_1, \dots, z_p)|$.

2 Existence of 2-bump solutions

In this section we consider the perturbed second order system of differential equations:

$$-\ddot{u} + u = \nabla V(u) + \epsilon \nabla_u W(t,u) \tag{2.1}$$

where $u \in \mathbb{R}^n$ $(n \geq 1)$.

2.1 Hypotheses and variational formulation

We assume that:

- (V_1) $V \in C^2(\mathbb{R}^n, \mathbb{R})$, $V(0) = 0$, $\nabla V(0) = 0$, $D^2 V(0) = 0$;
- (W_1) $W \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, $W(t,0) = 0$, $\nabla_u W(t,0) = 0$, $D^2_u W(t,0)$ is bounded and $D^2_u W(t,\cdot)$ is continuous uniformly with respect to t.

Because of (V_1) the origin is a hyperbolic equilibrium of the unperturbed system.
We will work in the Sobolev space $E = W^{1,2}(\mathbb{R}, \mathbb{R}^n)$. For $u \in E$ we set:

$$F(u) = -\int_{\mathbb{R}} V(u)dt \quad \text{and} \quad G(u) = -\int_{\mathbb{R}} W(t,u)dt.$$

Note that because of (V_1) and (W_1) the functionals F and G are well defined. Moreover, using the continuous embedding $W^{1,2}(\mathbb{R}, \mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}, \mathbb{R}^n)$, it can be checked that F and G are of class C^2 on E.

Homoclinic solutions of (2.1) are critical points of the functional:

$$f_\epsilon(u) = \frac{1}{2}||u||^2 + F(u) + \epsilon G(u). \tag{2.2}$$

As in [1] we require some non-degeneracy condition on the unperturbed equation:

$$-\ddot{u} + u = \nabla V(u). \tag{2.3}$$

We will assume:

- (V_2) $\exists u_0 \in E$ such that u_0 solves (2.3) and $Ker f_0''(u_0) = span\{\dot{u}_0\}$.

Then equation (2.3) has a homoclinic $u_0 \in E$ such that the solutions $\phi \in E$ of the linearized equation

$$-\ddot{\phi} + \phi = D^2 V(u_0(t))\phi \tag{2.4}$$

form a one dimensional space.

Since the functional f_0 is invariant under the action of $\mathbb{R}$, $Z = \{u_\theta = u_0(\cdot + \theta) \mid \theta \in \mathbb{R}\}$ is a C^2 one-dimensional manifold of critical points at level $b = f_0(u_0)$ and $T_{u_\theta} Z = span\{\dot{u}_\theta\} = \langle \dot{u}_\theta \rangle$.

Hypothesis (V_2) implies that the critical manifold Z is non-degenerate, *i.e.* $T_{u_\theta} Z = Ker f_0''(u_\theta)$ for any $\theta \in \mathbb{R}$.

2.2 A result on the existence of infinitely many 2-bump solutions

In this paragraph, for the sake of clarity, we show which kind of results can be obtained in section 2.

Let us consider the Melnikov primitive $\Gamma : \mathbb{R} \to \mathbb{R}$ defined by:

$$\Gamma(\theta) = -\int_{\mathbb{R}} W(t, u_0(t+\theta))dt.$$

We make the following hypotheses on Γ:

Condition 1 *There are $\eta > 0$ and a sequence $(U_n = (c_n, d_n))_{n\in\mathbb{Z}}$ of bounded open intervals of $\mathbb{R}$ which satisfy:*
(i) *$\Gamma_{|U_n}$ attains its minimum at some $a_n \in (c_n, d_n)$ and $\Gamma_{|\{c_n,d_n\}} \geq \Gamma(a_n) + \eta$;*
(ii) *$c_n \to +\infty$ as $n \to +\infty$ and $d_n \to -\infty$ as $n \to -\infty$.*

At the end of section 2 we will list some cases in which Condition 1 is satisfied.
The following theorem will be proved in section 2.5:

Theorem 1 *Let (V_1),(V_2),(W_1) and condition 1 hold. For $\epsilon \neq 0$, $|\epsilon|$ small enough there exists L_ϵ such that if $c_{i_2} - d_{i_1} > L_\epsilon$ then f_ϵ has a critical point u_ϵ located near some $u_{\theta_1} + u_{\theta_2}$ with $\theta_1 \in U_{i_1}$ and $\theta_2 \in U_{i_2}$.*

An immediate consequence of theorem 1 is the following corollary:

Corollary 1 *Let (V_1),(V_2),(W_1) and condition 1 hold. There exists $\epsilon_1 > 0$ such that $\forall|\epsilon| < \epsilon_1$, $\epsilon \neq 0$ equation (2.1) has infinitely many 2-bump solutions.*

Remark 1 *At the end of section 2 we will state a more general result (see theorem 2) on the existence of k-bump solutions.*

2.3 The natural constraint $Z_{L,\epsilon}$

Definition 1 *A submanifold $M \subset E$ is called a natural constraint for the functional f_ϵ if*

$$u \in M \text{ and } (f_{\epsilon|M})'(u) = 0 \text{ imply that } f'_\epsilon(u) = 0.$$

Under hypotheses (V_1),(V_2) and (W_1) Ambrosetti and Badiale in [1] build a 1-dimensional natural constraint Z_ϵ for f_ϵ near the critical manifold Z. Our aim is to build a natural constraint for 2-bump solutions.

Consider a cut-off function $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that:

$$\varphi(t) = 1 \text{ for } |t| < \frac{1}{8}, \quad \varphi(t) = 0 \text{ for } |t| > \frac{1}{4} \quad \text{and} \quad ||\dot\varphi||_\infty < 16.$$

Next define the function:

$$u_0^L(t) = \varphi(\frac{t}{L})u_0(t).$$

We will denote by u_θ^L the translate of u_0^L: $u_\theta^L = u_0^L(\theta + \cdot) = \theta * u_0^L$. Note that supp $u_0^L \subset [-\frac{L}{4}, \frac{L}{4}]$ and that if $\theta_2 - \theta_1 > L$ then supp $u_{\theta_1}^L \cap$ supp $u_{\theta_2}^L = \emptyset$. Moreover:

$$u_0^L \to u_0 \quad \text{and} \quad \dot u_0^L \to \dot u_0 \quad \text{as} \quad L \to +\infty.$$

(Explicit estimates on $||u_0 - u_0^L||$ and $||\dot{u}_0 - \dot{u}_0^L||$ will be given in lemma 10-(i)). In order to prove theorem 1 we will build for L large enough a natural constraint $Z_{L,\epsilon}$ for the functional f_ϵ close to the 2 dimensional manifold:

$$Z_L = \{u_{\theta_1}^L + u_{\theta_2}^L \mid \theta_2 - \theta_1 > L\}.$$

This will be possible because by (V_1):

$$f_0{}'(u_{\theta_1}^L + u_{\theta_2}^L) = f_0{}'(u_{\theta_1}^L) + f_0{}'(u_{\theta_2}^L) \to 0 \tag{2.5}$$

if $\theta_2 - \theta_1 > L$ and $L \to +\infty$.

In order to build the C^1 manifold $Z_{L,\epsilon}$ we need some lemmas.

We define the norm of $(X, \mu_1, \mu_2) \in E \times \mathbb{R}^2$ by $||(X, \mu_1, \mu_2)|| = ||X|| + |\mu_1| + |\mu_2|$.

The tangent space to Z_L at $u_{\theta_1}^L + u_{\theta_2}^L$ is equal to $\langle \dot{u}_{\theta_1}^L, \dot{u}_{\theta_2}^L \rangle$.

In the sequel we shall always assume that $L > 8$ (it implies in particular that (supp $u_{\theta_1}^L + [-2,2]) \cap ($supp $u_{\theta_2}^L + [-2,2]) = \emptyset$ if $\theta_2 - \theta_1 > L$).

For δ positive let:

$$V_L^\delta = \{(v, \theta_1, \theta_2) \in E \times \mathbb{R}^2 \mid \theta_2 - \theta_1 > L, v \in \langle \dot{u}_{\theta_1}^L, \dot{u}_{\theta_2}^L \rangle^\perp \text{ and } ||v|| < \delta\}.$$

Let $h^L : V_L^\delta \to E$ be the map defined by $h^L(v, \theta_1, \theta_2) = u_{\theta_1}^L + u_{\theta_2}^L + v$.

Lemma 1 *There is $\delta_0 > 0$ such that, if $L > 8$, h^L is a diffeomorphism from $V_L^{\delta_0}$ onto a neighborhood of Z_L.*

Proof:

For $\delta > 0$ we denote by B_δ the ball of center 0 and radius δ in E.

Let $\psi^L : B_\delta \times \{(\theta_1, \theta_2) \mid \theta_2 - \theta_1 > L\} \to E \times \mathbb{R}^2$ be defined by:

$$\psi^L(v, \theta_1, \theta_2) = (v + u_{\theta_1}^L + u_{\theta_2}^L, (v, \dot{u}_{\theta_1}^L), (v, \dot{u}_{\theta_2}^L)).$$

We shall prove that there exists δ_0 such that for any $L > 8$, ψ^L is a C^1 diffeomorphism onto a neighborhood of $Z_L \times \{0\}$ in $E \times \mathbb{R}^2$.

We will prove that for some $\delta_0 > 0$ and for any $L > 8$:

(i) ψ^L is a local diffeomorphism on $B_{\delta_0} \times \{(\theta_1, \theta_2) \mid \theta_2 - \theta_1 > L\}$.

(ii) ψ^L is injective on $B_{\delta_0} \times \{(\theta_1, \theta_2) \mid \theta_2 - \theta_1 > L\}$.

From **(i)** and **(ii)** we will have that ψ^L is a global diffeomorphism from $B_{\delta_0} \times \{(\theta_1, \theta_2) | \theta_2 - \theta_1 > L\}$ onto a neighborhood of $Z_L \times \{0\}$.

Finally we will get lemma 1 noticing that $V_L^{\delta_0} = (\psi^L)^{-1}(E \times \{0\})$.

Proof of (i):

Let $(v, \theta_1, \theta_2) \in B_\delta \times \{(\theta_1, \theta_2) | \theta_2 - \theta_1 > L\}$.

We have:

$$d_{(v,\theta_1,\theta_2)}\psi^L(X, \lambda_1, \lambda_2) = (X + \lambda_1 \dot{u}_{\theta_1}^L + \lambda_2 \dot{u}_{\theta_2}^L, (X, \dot{u}_{\theta_1}^L) + \lambda_1(v, \ddot{u}_{\theta_1}^L), (X, \dot{u}_{\theta_2}^L) + \lambda_2(v, \ddot{u}_{\theta_2}^L)).$$

Let B denote the linear operator defined on $E \times \mathbb{R}^2$ by:

$$B(X, \lambda_1, \lambda_2) = (X + \lambda_1 \dot{u}_{\theta_1}^L + \lambda_2 \dot{u}_{\theta_2}^L, (X, \dot{u}_{\theta_1}^L), (X, \dot{u}_{\theta_2}^L)).$$

We have:

$$||(d_{(v,\theta_1,\theta_2)}\psi^L - B)(X, \lambda_1, \lambda_2)|| \le \delta C ||(X, \lambda_1, \lambda_2)|| \tag{2.6}$$

where $C = \sup_{L>8} ||\ddot{u}_0^L|| < +\infty$.

Moreover for all $X \in E$ we can write $X = Y + \mu_1 \dot{u}_{\theta_1}^L + \mu_2 \dot{u}_{\theta_2}^L$ where $Y \in \langle \dot{u}_{\theta_1}^L, \dot{u}_{\theta_2}^L \rangle^\perp$, $||\dot{u}_0^L||^2 \mu_i = (X, \dot{u}_{\theta_i}^L)$ for i=1,2.

Thus

$$||B(X, \lambda_1, \lambda_2)|| = ||Y + (\mu_1 + \lambda_1)\dot{u}_{\theta_1}^L + (\mu_2 + \lambda_2)\dot{u}_{\theta_2}^L|| + ||\dot{u}_0^L||^2(|\mu_1| + |\mu_2|).$$

and, since $Y, \dot{u}_{\theta_1}^L, \dot{u}_{\theta_2}^L$ are pairwise orthogonal and $\inf_{L>8} ||\dot{u}_0^L|| > 0$, we get:

$$\begin{aligned} ||B(X, \lambda_1, \lambda_2)|| &\geq C'(||Y|| + |\mu_1 + \lambda_1| + |\mu_2 + \lambda_2| + |\mu_1| + |\mu_2| \\ &\geq C'(||Y|| + |\lambda_1|/2 + |\lambda_2|/2 + |\mu_1|/2 + |\mu_2|/2). \end{aligned}$$

Hence, since $||X|| \leq ||Y|| + (|\mu_1| + |\mu_2|)||\dot{u}_0^L||^2$ and $\max_{L>8} ||\dot{u}_0^L||^2 < \infty$, there exists $C'' > 0$ such that:

$$||B(X, \lambda_1, \lambda_2)|| \geq C''(||X|| + |\lambda_1| + |\lambda_2|). \tag{2.7}$$

Now choose $\delta_0 > 0$ such that $\delta_0 C < C''/2$. By (2.6) and (2.7) we have that for all $L > 8$, for all $(v, \theta_1, \theta_2) \in B_\delta \times \{(\theta_1, \theta_2) | \theta_2 - \theta_1 > L)\}$

$$||d_{(v,\theta_1,\theta_2)} \psi^L(X, \lambda_1, \lambda_2)|| \geq \frac{C''}{2} ||(X, \lambda_1, \lambda_2)||. \tag{2.8}$$

Thus $d_{(v,\theta_1,\theta_2)} \psi^L$ is injective; in addition this operator has the form "Id + compact" hence it is an isomorphism and ψ^L is a local diffeomorphism on $B_{\delta_0} \times \{(\theta_1, \theta_2) | \theta_2 - \theta_1 > L\}$.

Proof of (ii):

It is easy to check that for all $\gamma > 0$ there is $\eta > 0$ independent of L, such that for all (v, θ_1, θ_2), $(v', \theta_1', \theta_2')$ in $B_{\delta_0} \times \{(\theta_1, \theta_2) | \theta_2 - \theta_1 > L\}$

$$||v - v'|| + |\theta_1 - \theta_1'| + |\theta_2 - \theta_2'| < \eta \Longrightarrow ||d_{(v,\theta_1,\theta_2)} \psi^L - d_{(v',\theta_1',\theta_2')} \psi^L|| < \gamma.$$

This uniform continuity property, combined with (2.8), gives that there exists $\nu > 0$ (independent of L) such that:

$$0 < |\theta_1 - \theta_1'| + |\theta_2 - \theta_2'| + ||v - v'|| < \nu \Rightarrow \psi^L(v, \theta_1, \theta_2) \neq \psi^L(v', \theta_1', \theta_2'). \tag{2.9}$$

Now assume that : $\psi^L(\theta_1', \theta_2', v') = \psi^L(\theta_1, \theta_2, v)$. Then

$$||u_{\theta_1}^L + u_{\theta_2}^L - u_{\theta_1'}^L - u_{\theta_2'}^L||_\infty = ||v' - v||_\infty \leq ||v' - v|| \leq 2\delta_0. \tag{2.10}$$

Since $\theta_1 < \theta_2$ and $\theta_1' < \theta_2'$ we can assume that $\min(\theta_1, \theta_1', \theta_2, \theta_2') = \theta_1$.

Thus $\theta_2 - \theta_1 > L$ and $\theta_2' - \theta_1' > L$.

Hence $[-\theta_1 - L/4, -\theta_1 + L/4] \cap$ (supp $u_{\theta_2}^L$ $\cup$ supp $u_{\theta_2'}^L$) $= \emptyset$ and by (2.10)

$$||u_{\theta_1}^L - u_{\theta_1'}^L||_{L^\infty[-\theta_1 - L/4, -\theta_1 + L/4]} \leq 2\delta_0.$$

Now it is easy to see that there is $\mu > 0$, independent of $L > 8$, such that

$$||u_{\theta_1}^L - u_{\theta_1'}^L||_{L^\infty[-\theta_1 - L/4, -\theta_1 + L/4]} < \mu \Longrightarrow |\theta_1 - \theta_1'| < \nu/4. \tag{2.11}$$

Taking δ_0 smaller if necessary we can assume that $2\delta_0 < \mu$ and $2\delta_0 < \nu/2$.

Thus we get $||v - v'|| < \nu/2, |\theta_1 - \theta_1'| < \nu/4$ and (by the same way) $|\theta_2 - \theta_2'| < \nu/4$.

Therefore by (2.9) we get $(v,\theta_1,\theta_2)=(v',\theta'_1,\theta'_2)$. This concludes the proof of (ii). □

In [1] the manifold Z_ϵ is found by mean of the implicit function theorem applied to the map:

$$H \ : \ \mathbb{R}\times\mathbb{R}\times E\times\mathbb{R}\to E\times\mathbb{R}$$

with components H_1 and H_2 given by:

$$\begin{aligned} H_1(\epsilon,\sigma,v,\gamma) &= f'_\epsilon(u_\sigma+v)-\gamma\dot u_\sigma, \\ H_2(\epsilon,\sigma,v,\gamma) &= (v,\dot u_\sigma). \end{aligned}$$

The implicit function theorem can be applied because $Ker f''_0(u_\sigma)=\langle\dot u_\sigma\rangle$ implies that the partial derivative of H with respect to (v,γ) evaluated in $(0,\sigma,0,\gamma)$ is invertible.

We will generalize this approach to build $Z_{L,\epsilon}$.

Let us define the function:

$$H^L \ : \ \mathbb{R}\times\mathbb{R}^2\times E\times\mathbb{R}^2\to E\times\mathbb{R}^2$$

with components $H^L_1\in E$ and $H^L_2\in\mathbb{R}^2$ given by:

$$\begin{aligned} H^L_1(\epsilon,\theta_1,\theta_2,v,\alpha_1,\alpha_2) &= f'_\epsilon(u^L_{\theta_1}+u^L_{\theta_2}+v)-\alpha_1\dot u^L_{\theta_1}-\alpha_2\dot u^L_{\theta_2}, \\ H^L_2(\epsilon,\theta_1,\theta_2,v,\alpha_1,\alpha_2) &= ((v,\dot u^L_{\theta_1}),(v,\dot u^L_{\theta_2})). \end{aligned}$$

Consider the partial derivative of H^L:

$$\frac{\partial H^L}{\partial(v,\alpha)}=(\frac{\partial H^L_1}{\partial(v,\alpha)},\frac{\partial H^L_2}{\partial(v,\alpha)})$$

evaluated at $(0,\theta,0,\alpha)$ (where $\theta=(\theta_1,\theta_2)$ and $\alpha=(\alpha_1,\alpha_2)$). It's the linear operator of $E\times\mathbb{R}^2$ given by:

$$\begin{aligned} \frac{\partial H^L_1}{\partial(v,\alpha)}_{|(0,\theta,0,\alpha)}[X,\mu_1,\mu_2] &= f''_0(u^L_{\theta_1}+u^L_{\theta_2})X-\mu_1\dot u^L_{\theta_1}-\mu_2\dot u^L_{\theta_2}, \\ \frac{\partial H_2}{\partial(v,\alpha)}_{|(0,\theta,0,\alpha)}[X,\mu_1,\mu_2] &= ((X,\dot u^L_{\theta_1}),(X,\dot u^L_{\theta_2})). \end{aligned}$$

Since H^L is linear in (α_1,α_2) there results that $\frac{\partial H^L}{\partial(v,\alpha)}_{|(\epsilon,\theta,v,\alpha)}=\frac{\partial H^L}{\partial(v,\alpha)}_{|(\epsilon,\theta,v)}$ is independent of (α_1,α_2).

In lemma 2 we will prove that, provided L is great enough, $\frac{\partial H^L}{\partial(v,\alpha)}_{|(0,\theta,0)}$ is invertible and the norm of the inverse satisfies a uniform bound.

In lemmas 3 and 4 we will show how to build $Z_{L,\epsilon}$.

Lemma 2 *There exist positive constants $C_1,L_1>8$ such that $\forall L>L_1$, $\forall\theta=(\theta_1,\theta_2)$ with $\theta_2-\theta_1>L$, and $\forall(X,\mu_1,\mu_2)\in E\times\mathbb{R}^2$:*

$$||\frac{\partial H^L}{\partial(v,\alpha)}_{|(0,\theta,0)}[X,\mu_1,\mu_2]||\geq C_1||(X,\mu_1,\mu_2)||, \tag{2.12}$$

i.e :

$$||f_0''(u^L_{\theta_1}+u^L_{\theta_2})X-\mu_1\dot u^L_{\theta_1}-\mu_2\dot u^L_{\theta_2}||+|(X,\dot u^L_{\theta_1})|+|(X,\dot u^L_{\theta_2})|\geq C_1(||X||+|\mu_1|+|\mu_2|). \tag{2.13}$$

Proof:

Arguing by contradiction we assume that the statement in lemma 2 does not hold.

Then we can define sequences $L_n, \theta^n = (\theta_1^n, \theta_2^n)$, $(X_n, \mu_1^n, \mu_2^n) \in E \times \mathbb{R}^2$ such that $||(X_n, \mu_1^n, \mu_2^n)|| = 1$, $L_n \to +\infty$, $\theta_2^n - \theta_1^n > L_n$ and

$$||f_0{}''(u_{\theta_1^n}^{L_n} + u_{\theta_2^n}^{L_n})X_n - \mu_1^n \dot{u}_{\theta_1^n}^{L_n} - \mu_2^n \dot{u}_{\theta_2^n}^{L_n}|| \to 0, \tag{2.14}$$

$$|(X_n, \dot{u}_{\theta_1^n}^{L_n})| + |(X_n, \dot{u}_{\theta_2^n}^{L_n})| \to 0. \tag{2.15}$$

Using the invariance of f_0 and of the scalar product $(\cdot, \cdot)$ under the action of $\mathbb{R}$ we can assume that $\theta_1^n = 0$ for any n.

Since $||X_n||, |\mu_1^n|, |\mu_2^n|$ are bounded, up to a subsequence $X_n \rightharpoonup X$, $\mu_1^n \to \mu_1, \mu_2^n \to \mu_2$. We are going to show that $X = 0, \mu_1 = 0$.

Let $g \in E$ be fixed. From (2.14) we get that:

$$(f_0{}''(u_0^{L_n} + u_{\theta_2^n}^{L_n})X_n, g) - \mu_1^n(\dot{u}_0^{L_n}, g) - \mu_2^n(\dot{u}_{\theta_2^n}^{L_n}, g) \to 0. \tag{2.16}$$

Since $D^2V(0) = 0$ we have

$$\begin{aligned}(f_0{}''(u_0^{L_n} + u_{\theta_2^n}^{L_n})X_n, g) &= (X_n, g) - \int_{\mathbb{R}} D^2V(u_0^{L_n} + u_{\theta_2^n}^{L_n})X_n g \\ &= (X_n, g) - \int_{\mathbb{R}} D^2V(u_0^{L_n})X_n g - \int_{\mathbb{R}} D^2V(u_{\theta_2^n}^{L_n})X_n g.\end{aligned}$$

Since $u_0^{L_n} \to u_0$ in L^∞, by the uniform continuity of D^2V on bounded subsets of $\mathbb{R}^n$, $D^2V(u_0^{L_n}) \to D^2V(u_0)$ in L^∞. Consequently,

$$\int_{\mathbb{R}} D^2V(u_0^{L_n})X_n g \to \int_{\mathbb{R}} D^2V(u_0)Xg. \tag{2.17}$$

Next supp $u_{\theta_2^n}^{L_n} \subset [-\theta_2^n - L_n/4, -\theta_2^n + L_n/4]$ and $\theta_2^n > L_n$ hence supp $u_{\theta_2^n}^{L_n} \subset (-\infty, -3L_n/4)$. Therefore, since $D^2V(0) = 0$,

$$\left|\int_{\mathbb{R}} D^2V(u_{\theta_2^n}^{L_n})X_n g\right| = \left|\int_{-\infty}^{-3L_n/4} D^2V(u_{\theta_2^n}^{L_n})X_n g\right| \le ||D^2V(u_{\theta_2^n}^{L_n})||_\infty ||X_n||_2 ||g||_{L^2(-\infty,-3L_n/4)}.$$

$||X_n||_2$ and $||D^2V(u_{\theta_2^n}^{L_n})||_\infty$ being bounded we get

$$\int_{\mathbb{R}} D^2V(u_{\theta_2^n}^{L_n})X_n g \to 0. \tag{2.18}$$

as $n \to \infty$. Moreover $X_n \rightharpoonup X$, $\dot{u}_0^{L_n} \to \dot{u}_0$ and $\dot{u}_{\theta_2^n}^{L_n} \rightharpoonup 0$ imply

$$(X_n, g) \to (X, g), \qquad \mu_1^n(\dot{u}_0^{L_n}, g) \to \mu_1(\dot{u}_0, g), \qquad \mu_2^n(\dot{u}_{\theta_2^n}^{L^n}, g) \to 0. \tag{2.19}$$

From (2.16),(2.17),(2.18) and (2.19) we get

$$(X, g) + \int_{\mathbb{R}} D^2V(u_0)Xg - \mu_1(\dot{u}_0, g) = 0.$$

Since this equality holds for all $g \in E$ we have that:

$$f_0''(u_0)X = \mu_1 \dot{u}_0. \tag{2.20}$$

Since $f_0''(u_0)$ is symmetric and $\dot{u}_0 \in \ker f_0''(u_0)$ (2.20) implies that $\mu_1 = 0$ and $X \in \ker f_0''(u_0) = TZ_{u_0} = \mathbb{R}\dot{u}_0$. Now from

$$(X_n, \dot{u}_0^{L_n}) \to 0$$

we have $(X, \dot{u}_0) = 0$. Hence $X = 0$.

Thus $X_n \rightharpoonup 0$ and $\mu_1^n \to 0$. Similarly it can be proved that $(-\theta_2^n) * X_n \rightharpoonup 0$ and $\mu_2 = 0$.

Now, from (2.14), since μ_1^n and $\mu_2^n \to 0$ we have:

$$||X_n||^2 - \int_{\mathbb{R}} D^2V(u_0^{L_n})X_n^2 - \int_{\mathbb{R}} D^2V(u_{\theta_2^n}^{L_n})X_n^2 \to 0. \tag{2.21}$$

We can write

$$\int_{\mathbb{R}} D^2V(u_0^{L_n})X_n^2 = \int_{-A}^{A} D^2V(u_0^{L_n})X_n^2 + \int_{[-A,A]^c} D^2V(u_0^{L_n})X_n^2.$$

Since $||u_0^{L_n}||_{L^\infty([-A,A]^c)} \to 0$ as $A \to \infty$ independently of n and $D^2V(0) = 0$, the latter term in the sum tends to 0 as $A \to \infty$ independently of n. Moreover, for A fixed, the former term tends to 0 because $||X_n||_{L^2([-A,A])} \to 0$. Hence

$$\int_{\mathbb{R}} D^2V(u_0^{L_n})X_n^2 \to 0.$$

Similarly, using that $(-\theta_2^n) * X_n \to 0$ we get

$$\int_{\mathbb{R}} D^2V(u_{\theta_2^n}^{L_n})X_n^2 \to 0.$$

These properties and (2.21) imply that $||X_n|| \to 0$, hence $||X_n|| + |\mu_1^n| + |\mu_2^n| \to 0$, which contradicts $||X_n|| + |\mu_1^n| + |\mu_2^n| = 1$. □.

Remark 2 *It is easy to see that f_0'' is uniformly continuous on bounded subsets of E. Hence there exist positive constants $\delta_1 \le \delta_0, C_2 > 0$ such that for all $L > L_1$, for all $\theta = (\theta_1, \theta_2)$ such that $\theta_2 - \theta_1 > L$ for all $v \in E$ with $||v|| \le \delta_1$*

$$||\frac{\partial H^L}{\partial(v,\alpha)}_{|(0,\theta,v)}[X, \mu_1, \mu_2]|| \ge C_2 ||(X, \mu_1, \mu_2)||.$$

Since $\frac{\partial H^L}{\partial(v,\alpha)}_{|(0,\theta,v)}$ is of the form Id + compact, the latter estimate implies that it is invertible.

In order to build a natural constraint $Z_{L,\epsilon}$ for f_ϵ, close to Z_L, we first build a natural constraint M_L, diffeomorphic to Z_L, for the unperturbed functional f_0.

We can prove that:

Lemma 3 *There exists $L_2 \ge L_1$ such that $\forall L > L_2$ there exists a C^1 function $w(L,\cdot) : \{(\theta_1, \theta_2) \mid \theta_2 - \theta_1 > L\} \to \{v \in E \mid ||v|| < \delta_1\}$ such that $w(L, \theta_1, \theta_2) \in \langle \dot{u}_{\theta_1}^L, \dot{u}_{\theta_2}^L \rangle^\perp$ and*

$$f_0'(u_{\theta_1}^L + u_{\theta_2}^L + w(L, \theta_1, \theta_2)) = \alpha_1 \dot{u}_{\theta_1}^L + \alpha_2 \dot{u}_{\theta_2}^L \tag{2.22}$$

for some $(\alpha_1, \alpha_2) \in \mathbb{R}^2$.

Moreover $\sup_{\{(\theta_1,\theta_2) | \theta_2 - \theta_1 > L\}} ||w(L, \theta_1, \theta_2)|| \to 0$ as $L \to +\infty$.

Proof:
In this proof we shall use the following abbreviation:

$$b^L(\theta,w) = \frac{\partial H^L}{\partial(v,\alpha)}_{|(0,\theta,w)} \quad \in \; L(E\times \mathbb{R}^2)$$

We have to find w, α_1, α_2 such that

$$H^L(0,\theta_1,\theta_2,w,\alpha_1,\alpha_2) = 0. \tag{2.23}$$

In fact $H_1^L = 0$ means that $f_0'(u_{\theta_1}^L + u_{\theta_2}^L + w(L,\theta_1,\theta_2)) = \alpha_1 \dot{u}_{\theta_1}^L + \alpha_2 \dot{u}_{\theta_2}^L$ and $H_2^L = 0$ means that $w \in \langle \dot{u}_{\theta_1}^L, \dot{u}_{\theta_1}^L \rangle^\perp$.

Let $B_\delta \subset E \times \mathbb{R}^2$ be the ball in $E\times\mathbb{R}^2$ of center 0 and radius δ: $B_\delta = \{(w,\alpha_1,\alpha_2) \mid ||w|| + |\alpha_1| + |\alpha_2| \le \delta\}$. By lemma 2 and remark 2,

$$\forall L > L_1,\ \forall(\theta_1,\theta_2) \text{ such that } \theta_2-\theta_1 > L,\ ||(b^L)^{-1}(\theta_1,\theta_2,0)|| \le \frac{1}{C_1}. \tag{2.24}$$

In order to solve equation (2.23) we do not apply directly the implicit function theorem because $H^L(0,\theta_1,\theta_2,0,0,0) \neq 0$; we only have by (2.5) that $H^L(0,\theta_1,\theta_2,0,0,0) \to 0$ if $\theta_2 - \theta_1 > L$ and $L \to +\infty$. We will solve equation (2.23) by means of the contraction-mapping theorem, proving that, provided L is large enough and δ small enough, for all (θ_1,θ_2) with $\theta_2 - \theta_1 > L$ there is a unique $(w(\theta_1,\theta_2), \alpha_1(\theta_1,\theta_2), \alpha_2(\theta_1,\theta_2)) \in B_\delta$ such that $H^L(0,\theta_1,\theta_2,w(\theta_1,\theta_2),\alpha_1(\theta_1,\theta_2),\alpha_2(\theta_1,\theta_2)) = 0$.

Indeed $H^L(0,\theta_1,\theta_2,w,\alpha_1,\alpha_2) = 0$ is equivalent to $R(w,\alpha_1,\alpha_2) = (w,\alpha_1,\alpha_2)$ where:

$$R(w,\alpha) = -(b^L)^{-1}(\theta,0)H^L(0,\theta,0,0) - (b^L)^{-1}(\theta,0)(H^L(0,\theta,w,\alpha) - H^L(0,\theta,0,0) - b^L(\theta,0)[w,\alpha]).$$

We will find L_2 and $\delta < \delta_1$ such that if $L > L_2$ and $\theta_2 - \theta_1 > L$ then

- **(i)** $R(B_\delta) \subset B_{\frac{2\delta}{3}}$;
- **(ii)** R is a contraction on B_δ, more precisely:

$$\forall (w,\alpha),(w',\alpha') \in B_\delta \quad ||R(w,\alpha) - R(w',\alpha')|| \le \frac{1}{3}||(w,\alpha)-(w',\alpha)||.$$

Since f_0'' is uniformly continuous on bounded subsets of E, we can choose $0 < \delta < \delta_1$ such that:

$$\forall L > L_1, \forall\theta = (\theta_1,\theta_2) \text{ with } \theta_2-\theta_1 > L \text{ and } \forall w \text{ with } ||w|| \le \delta,\ ||b^L(\theta,w) - b^L(\theta,0)|| < \frac{C_1}{3}. \tag{2.25}$$

Moreover by (2.5) there is $L_2 \ge L_1$ such that:

$$\forall L > L_2 \text{ and } \forall \theta = (\theta_1,\theta_2) \in \mathbb{R}^2 \text{with } \theta_2 - \theta_1 > L,\ ||H^L(0,\theta,0,0)|| < \frac{C_1\delta}{3}. \tag{2.26}$$

Let $L > L_2$ and (θ_1,θ_2) with $\theta_2 - \theta_1 > L$ be fixed.

Using (2.24), (2.25) and (2.26) we now prove (i). $\forall (w,\alpha_1,\alpha_2) \in B_\delta$ one has (setting $\alpha = (\alpha_1,\alpha_2)$):

$$\begin{aligned} ||R(w,\alpha)|| \;&\le\; ||-(b^L)^{-1}(\theta,0)H^L(0,\theta,0,0)|| \\ &+\; ||(b^L)^{-1}(\theta,0)|| \cdot ||H^L(0,\theta,w,\alpha) - H^L(0,\theta,0,0) - b^L(\theta,0)[w,\alpha]|| \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{C_1}\frac{C_1\delta}{3} + \frac{1}{C_1}||H^L(0,\theta,w,\alpha) - H^L(0,\theta,0,0) - b(\theta,0)[w,\alpha]|| \\
&= \frac{\delta}{3} + \frac{1}{C_1}||\int_0^1 (b^L(\theta_1,\theta_2,sw) - b^L(\theta_1,\theta_2,0))[w,\alpha_1,\alpha_2]ds|| \\
&\le \frac{\delta}{3} + \frac{1}{C_1}\int_0^1 ||b^L(\theta_1,\theta_2,sw) - b^L(\theta_1,\theta_2,0)||\delta ds \\
&\le \frac{\delta}{3} + \frac{1}{C_1}\int_0^1 \frac{C_1}{3}\delta ds = \frac{2\delta}{3}.
\end{aligned}
$$

Still by (2.24),(2,25) and (2.26) we prove (ii): $\forall (w,\alpha_1,\alpha_2),(w',\alpha_1',\alpha_2') \in B_\delta$ one has:

$$
\begin{aligned}
||R(w,\alpha) - R(w',\alpha')|| &= ||(b^L)^{-1}(\theta,0)\left(H^L(0,\theta,w,\alpha) - H^L(0,\theta,w',\alpha') - b^L(\theta,0)[(w,\alpha)-(w',\alpha')]\right)|| \\
&\le \frac{1}{C_1}||\int_0^1 (b^L(\theta,w+s(w'-w)) - b^L(\theta,0))[(w,\alpha)-(w',\alpha')]ds|| \\
&\le \frac{1}{C_1}\int_0^1 \frac{C_1}{3}||(w,\alpha)-(w',\alpha')||ds = \frac{1}{3}||(w,\alpha)-(w',\alpha')||.
\end{aligned}
$$

Applying the contraction-mapping theorem we conclude that there is a unique $(w(L,\theta_1,\theta_2)$, $\alpha_1(L,\theta_1,\theta_2)$, $\alpha_2(L,\theta_1,\theta_2)) \in B_\delta$ such that $H^L(0,\theta_1,\theta_2,w(L,\theta_1,\theta_2),\alpha_1(L,\theta_1,\theta_2),\alpha_2(L,\theta_1,\theta_2)) = 0$.

Since $H^L(0,\theta_1,\theta_2,0,0) \to 0$ as $\theta_2-\theta_1 > L$ and $L \to +\infty$, by the properties of R we get that $w(L,\theta_1,\theta_2) \to 0$ as $\theta_2-\theta_1 > L$ and $L \to \infty$.

We now justify that $w(L,\cdot)$ is C^1. Indeed

$$H^L(0,\theta_1,\theta_2,w(L,\theta_1,\theta_2),\alpha_1(L,\theta_1,\theta_2),\alpha_2(L,\theta_1,\theta_2)) = 0,$$

$H^L(0,\cdot)$ is a C^1 function of (θ,w,α) and for $||(w,\alpha_1,\alpha_2)|| < \delta$

$$b(\theta,w) = \frac{\partial H^L}{\partial(v,\alpha)}_{|(0,\theta,w)} \in L(E\times \mathbb{R}^2)$$

is invertible. Hence the implicit function theorem can be applied and $w(L,\cdot)$ is C^1.□

Remark 3 *Since $w(L,\cdot)$ is C^1, $w \in \langle \dot u^L_{\theta_1}, \dot u^L_{\theta_2}\rangle^\perp$ and $||w|| < \delta_0$ we derive by lemma* 1 *that $M_L = \{u^L_{\theta_1} + u^L_{\theta_2} + w(L,\theta_1,\theta_2) \mid \theta_2-\theta_1 > L_2\}$ is a C^1 submanifold of E of dimension* 2 *and its tangent space at $u^L_{\theta_1} + u^L_{\theta_2} + w(L,\theta_1,\theta_2)$ is transversal to $\langle \dot u^L_{\theta_1}, \dot u^L_{\theta_2}\rangle^\perp$. This implies that M_L is a natural constraint for f_0 (see [1]).*

Now we can state another lemma which enables us to build the natural constraint $Z_{L,\epsilon}$ for f_ϵ.

Lemma 4 *There exists $\epsilon_0 > 0$ such that $\forall L > L_2$ there is a C^1 function $\bar w(L,\cdot) : (-\epsilon_0,\epsilon_0) \times \{(\theta_1,\theta_2) \in \mathbb{R}^2 \mid \theta_2-\theta_1 > L_2\} \to \{v \in E \mid ||v|| < \delta_1\}$ such that:*

- *$\bar w(L,0,\theta_1,\theta_2) = 0$;*
- *$f'_\epsilon(u^L_{\theta_1} + u^L_{\theta_2} + w(L,\theta_1,\theta_2) + \bar w(L,\epsilon,\theta_1,\theta_2)) \in \langle \dot u^L_{\theta_1}, \dot u^L_{\theta_2}\rangle$;*
- *$\bar w(L,\epsilon,\theta_1,\theta_2) \in \langle \dot u^L_{\theta_1}, \dot u^L_{\theta_2}\rangle^\perp$.*

Moreover there exists a positive constant C_3 such that $||\bar{w}(L,\epsilon,\theta)|| < C_3|\epsilon|$ for all $L > L_2$ and $\theta = (\theta_1,\theta_2)$ such that $\theta_2 - \theta_1 > L$.

We will also use the notation: $\bar{w}(L,\epsilon,\theta_1,\theta_2) = \bar{w}_\epsilon(L,\theta_1,\theta_2)$.

Proof: The proof is very similar to that of lemma 3. We don't use directly as in [1] the implicit function theorem because we have to justify that for any $\epsilon \in (-\epsilon_0,\epsilon_0)$ the function $\bar{w}(L,\epsilon,\cdot)$ is defined and C^1 on the whole set $\{(\theta_1,\theta_2) \in \mathbb{R}^2 \mid \theta_2 - \theta_1 > L\}$. We can apply the contraction mapping theorem uniformly on $\{(\theta_1,\theta_2) \in \mathbb{R}^2 \mid \theta_2 - \theta_1 > L_2\}$ because G'' is bounded on bounded subsets of E. $\square$

Finally we define for $L > L_2$ and $|\epsilon| < \epsilon_0$:

$$Z_{L,\epsilon} = \{u^L_{\theta_1} + u^L_{\theta_2} + w(L,\theta_1,\theta_2) + \bar{w}(L,\epsilon,\theta_1,\theta_2) \mid \theta_2 - \theta_1 > L\}.$$

By lemma 1, $Z_{L,\epsilon}$ is a C^1 2-dimensional submanifold of E and its tangent space at $u^L_{\theta_1} + u^L_{\theta_2} + w(L,\theta_1,\theta_2) + \bar{w}_\epsilon(L,\theta_1,\theta_2)$ is transversal to $\langle \dot{u}^L_{\theta_1}, \dot{u}^L_{\theta_2}\rangle^\perp$.

Hence, in the same way as in [1], it is easy to prove that:

Lemma 5 *$Z_{L,\epsilon}$ is a natural constraint for f_ϵ.*

Remark 4 *In the previous arguments we could have considered a more general perturbation term such as: $W(\epsilon,t,u) = \epsilon W(t,u) + o(\epsilon)W_1(\epsilon,t,u)$ where W and W_1 satisfy hypotheses (W_1). (See also [1]-[2]).*

2.4 Expression of f_ϵ on $Z_{L,\epsilon}$

By lemma 5 we are led, in order to find 2-bump solutions, to look for critical points of the functional f_ϵ restricted to the 2-dimensional manifold $Z_{L,\epsilon}$.

In the next lemma we find a suitable expression for the functional f_ϵ restricted to $Z_{L,\epsilon}$:

Lemma 6 *For $L > L_2$ and $|\epsilon| < \epsilon_0$, $f_{\epsilon|Z_{L,\epsilon}}$ has the following form:*

$$f_\epsilon(u^L_{\theta_1} + u^L_{\theta_2} + w(L,\theta_1,\theta_2) + \bar{w}_\epsilon(L,\theta_1,\theta_2)) = 2b + \epsilon(G(u_{\theta_1}) + G(u_{\theta_2})) + o_L(1) + O(\epsilon^2). \quad (2.27)$$

Proof:

Let $L > L_2$, $|\epsilon| < \epsilon_0$ and $\theta_2 - \theta_1 > L$. Since $\bar{w}_\epsilon \in \langle \dot{u}^L_{\theta_1}, \dot{u}^L_{\theta_2}\rangle^\perp$, by lemma 3, $(f_0'(u^L_{\theta_1} + u^L_{\theta_2} + w(L,\theta_1,\theta_2)), \bar{w}_\epsilon) = 0$; by lemma 4 $||\bar{w}_\epsilon|| \le C_3|\epsilon|$. Moreover, since f_0'' and G' are bounded on bounded subsets of E we can write:

$$\begin{aligned} f_\epsilon(u^L_{\theta_1} + u^L_{\theta_2} + w(L,\theta_1,\theta_2) + \bar{w}_\epsilon(L,\theta_1,\theta_2)) &= f_0(u^L_{\theta_1} + u^L_{\theta_2} + w + \bar{w}_\epsilon) + \epsilon G(u^L_{\theta_1} + u^L_{\theta_2} + w + \bar{w}_\epsilon) \\ &= f_0(u^L_{\theta_1} + u^L_{\theta_2} + w) + (f_0'(u^L_{\theta_1} + u^L_{\theta_2} + w), \bar{w}_\epsilon) \\ &+ O(||\bar{w}_\epsilon||^2) + \epsilon G(u^L_{\theta_1} + u^L_{\theta_2} + w) + \epsilon O(||\bar{w}_\epsilon||) \\ &= f_0(u^L_{\theta_1} + u^L_{\theta_2} + w) + \epsilon G(u^L_{\theta_1} + u^L_{\theta_2} + w) + O(\epsilon^2). \end{aligned}$$

There results that:

$$\begin{aligned} f_\epsilon(u^L_{\theta_1} + u^L_{\theta_2} + w(L,\theta_1,\theta_2) + \bar{w}_\epsilon(\epsilon,\theta_1,\theta_2)) &= 2b + \epsilon(G(u_{\theta_1}) + G(u_{\theta_2})) + \\ (f_0(u^L_{\theta_1} + u^L_{\theta_2} + w) - f_0(u_{\theta_1}) - f_0(u_{\theta_2})) &+ \epsilon(G(u^L_{\theta_1} + u^L_{\theta_2} + w) - G(u_{\theta_1}) - G(u_{\theta_2})) + O(\epsilon^2). \end{aligned}$$

Now, by lemma 3 $||w(L,\theta_1,\theta_2)|| = o_L(1)$ hence:

$$f_0(u^L_{\theta_1} + u^L_{\theta_2} + w(L,\theta_1,\theta_2)) = f_0(u^L_{\theta_1} + u^L_{\theta_2}) + o_L(1) = f_0(u^L_{\theta_1}) + f_0(u^L_{\theta_2}) + o_L(1).$$

In the same way

$$G(u^L_{\theta_1} + u^L_{\theta_2} + w(L, \theta_1, \theta_2)) = G(u^L_{\theta_1}) + G(u^L_{\theta_2}) + o_L(1).$$

Moreover

$$f_0(u^L_{\theta_1}) + f_0(u^L_{\theta_2}) - 2b = f_0(u^L_{\theta_1}) + f_0(u^L_{\theta_2}) - f_0(u_{\theta_1}) - f_0(u_{\theta_2}) = o_L(1)$$

and

$$G(u^L_{\theta_1}) + G(u^L_{\theta_2}) - G(u_{\theta_1}) - G(u_{\theta_2}) = o_L(1).$$

So we get formula (2.27) .□

It turns out that $G(u_\theta)$ is nothing but the Melnikov primitive:

$$G(u_\theta) = \Gamma(\theta) = -\int_{\mathbb{R}} W(t, u_0(t + \theta))dt. \tag{2.28}$$

2.5 Existence of infinitely many homoclinic solutions

This paragraph is devoted to the proof of theorem 1 and to a more general result on the existence of infinitely many homoclinic solutions of (2.1).

Proof of theorem 1:

Choose $0 < \epsilon_1 < \epsilon_0$ so small that in formula (2.27) $|O(\epsilon^2)| < |\epsilon|\eta/6$ for $|\epsilon| < \epsilon_1$, $\epsilon \neq 0$. Let $\epsilon \in (-\epsilon_1, 0) \cup (0, \epsilon_1)$ be fixed. Choose $L_\epsilon \geq L_2$ such that $\forall L > L_\epsilon$ one has in formula (2.27) $|o_L(1)| < |\epsilon|\eta/6$. We denote by $\bar{f}_\epsilon$ the function of 2 real variables (θ_1, θ_2) defined by $\bar{f}_\epsilon(\theta_1, \theta_2) = f_\epsilon(u^L_{\theta_1} + u^L_{\theta_2} + w(L, \theta_1, \theta_2) + \bar{w}(L, \epsilon, \theta_1, \theta_2))$.

Since $Z_{L,\epsilon}$ is a natural constraint for f_ϵ each critical point (θ_1, θ_2) of $\bar{f}_\epsilon$ provides a critical point u_ϵ of f_ϵ given by $u_\epsilon = u^L_{\theta_1} + u^L_{\theta_2} + w(L, \theta_1, \theta_2) + \bar{w}_\epsilon(L, \theta_1, \theta_2)$. Since $u^L_{\theta_i} \to u_{\theta_i}$ as $L \to +\infty$, $\sup_{\{\theta_2 - \theta_1 > L\}} ||w(L, \theta_1, \theta_2)|| \to 0$ as $L \to +\infty$ and $||\bar{w}_\epsilon(L, \theta_1, \theta_2)|| < |\epsilon|C_1$, the critical point u_ϵ is located near $u_{\theta_1} + u_{\theta_2}$. So it is enough to verify that a local minimum (if $\epsilon > 0$, otherwise a local maximum if $\epsilon < 0$) of $\bar{f}_\epsilon$ can be found in $U_{i_1} \times U_{i_2}$ for each $(i_1, i_2) \in \mathbb{Z}^2$ such that $c_{i_2} - d_{i_1} > L_\epsilon$. We make the proof for $\epsilon > 0$.

By formula (2.27) and our choice of ϵ_1 and L_ϵ, we have that:

$$\bar{f}_\epsilon(a_{i_1}, a_{i_2}) \leq 2b + \epsilon(\Gamma(a_{i_1}) + \Gamma(a_{i_2})) + \epsilon\frac{\eta}{3}.$$

Moreover $\forall(\theta_1, \theta_2) \in \partial(U_{i_1} \times U_{i_2})$ we have from condition 1 that:

$$\Gamma(\theta_1) + \Gamma(\theta_2) \geq \Gamma(a_{i_1}) + \Gamma(a_{i_2}) + \eta.$$

Therefore, using once more formula (2.27), if $(\theta_1, \theta_2) \in \partial(U_{i_1} \times U_{i_2})$ we have:

$$\bar{f}_\epsilon(\theta_1, \theta_2) \geq 2b + \epsilon(\Gamma(\theta_1) + \Gamma(\theta_2)) - \epsilon\frac{\eta}{3}$$

$$\geq 2b + \epsilon(\Gamma(a_{i_1}) + \Gamma(a_{i_2})) + \frac{2}{3}\epsilon\eta > \bar{f}_\epsilon(a_{i_1}, a_{i_2}).$$

The latter inequalities imply that $\bar{f}_\epsilon$ attains its minimum at some point (θ_1, θ_2) in $U_{i_1} \times U_{i_2}$. The proof of theorem 1 is complete.□

Remark 5 *Note that $L_\epsilon \to +\infty$ as $\epsilon \to 0$. In section 3, using the exponential decay property of u_0, we will show that it is possible to take $L_\epsilon = -K \ln |\epsilon|$ for some positive constant K.*

Remark 6 *It is also possible to obtain solutions* u_ϵ *of (2.1) located near* $u_{\theta_1} + u_{\theta_2}$ *where* θ_1 *and* θ_2 *are two maxima of* Γ.

Now we give some examples of perturbations W such that the Melnikov primitive $\Gamma : \mathbb{R} \to \mathbb{R}$ satisfies condition 1.

- **Periodic perturbation:**

  If the perturbation term $W(t, u)$ is T-periodic in time then Γ is T-periodic. Any non-constant periodic function Γ satisfies condition 1.

- **Quasi-periodic perturbation:**

  If $W(t, u)$ is quasi-periodic in time then Γ is quasi-periodic too. It is easy to show that a non-constant quasi-periodic function satisfies condition 1.

- **Almost-periodic perturbation:**

  Finally let us consider a perturbation term $W(t, u)$ almost-periodic in t uniformly with respect to x in compact subsets of $\mathbb{R}^n$.

  Let $H(W) \subset C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ be the closure, for the topology of uniform convergence on compact sets, of the set $\{W(\cdot + \tau, x) \mid \tau \in \mathbb{R}\}$.

  Bochner's criterion state that a function $W(t, x)$ is almost-periodic in t uniformly with respect to x in compact sets if and only if $H(W)$ is compact in $C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ for the topology of uniform convergence on sets $\mathbb{R} \times K$ with K compact.

  Using Bochner's criterion it is easy to see that if the perturbation $W(t, u)$ is almost-periodic in t uniformly with respect to x in compact sets then the Melnikov primitive Γ is almost-periodic.

  Therefore $\Gamma : \mathbb{R} \to \mathbb{R}$ satisfies condition 1, provided it is non constant.

It is straightforward to generalize the above construction to prove the existence of k-bump solutions as stated in the following theorem:

Theorem 2 *Let condition* (V_1), (V_2), (W_1) *and condition* (1) *hold. For all* k, *for* $\epsilon \neq 0$ *small enough there exists* L_ϵ *such that if* $\min_{l=1,\ldots,k-1}(c_{i_{l+1}} - d_{i_l}) > L_\epsilon$ *then* f_ϵ *has a critical point* u_ϵ *located near some* $u_{\theta_1} + \ldots + u_{\theta_k}$ *with* $\theta_l \in U_{i_l}$ *for* $l = 1, \ldots, k$.

As a consequence of theorem 2 we have the following corollary:

Corollary 2 *For all* k *there exists* $\overline{\epsilon} > 0$ *such that* $\forall \epsilon \in (-\overline{\epsilon}, 0) \cup (0, \overline{\epsilon})$ *equation* (2.1) *has infinitely many* k*-bump solutions.*

However the constants L_ϵ and $\overline{\epsilon}$ given by theorem 2 can depend on the number of bumps k so that theorem 2 cannot be directly used to obtain the existence of solutions with infinitely many bumps. The bound that we obtain for $||u_\epsilon - \sum_{i=1}^k u_{\theta_i}||$ is not independent of the number of bumps k. We will show in the next section how to derive estimates independent of k by using a different norm. We shall find solutions u_ϵ close to $\sum u_{\theta_i}$ only in L^∞-norm but not in H^1-norm. See also [17].

3 Existence of solutions with infinitely many bumps

In this section we show how to modify the previous lemmas in order to obtain constants $\bar{\epsilon}$ and L_ϵ independent of the number of bumps k.

In the sequel the symbol θ will mean $\theta = (\theta_1, \dots, \theta_k) \in \mathbb{R}^k$. We consider the manifold

$$Z_L^k = \{u_{\theta_1}^L + \dots + u_{\theta_k}^L \mid \min_i(\theta_{i+1} - \theta_i) > L\}.$$

Its tangent space at $u_{\theta_1}^L + \dots + u_{\theta_k}^L$ is $\langle \dot{u}_{\theta_1}^L, \dots, \dot{u}_{\theta_k}^L \rangle$.

For any $\theta_1 < \dots < \theta_k$ we will consider the norm on E:

$$|u|_\theta^2 = \max_{i=1,\dots,k} \int_{I_i} |u|^2 + |\dot{u}|^2$$

where

$I_1 = (\dfrac{-\theta_1 - \theta_2}{2}, +\infty), I_i = (\dfrac{-\theta_{i+1} - \theta_i}{2}, \dfrac{-\theta_i - \theta_{i-1}}{2})$ and $I_k = (-\infty, \dfrac{-\theta_k - \theta_{k-1}}{2})$. In the sequel $||\cdot||$ will still denote the H^1-norm.

Since for every $u \in E$ we have

$$|u|_\theta^2 \le ||u||^2 \le k|u|_\theta^2,$$

the norm $|\cdot|_\theta$ is equivalent to the H^1-norm for fixed k. Moreover the following uniform bound can be easily proved : $\forall k \in \mathbb{N}$, $\forall (\theta_1, ..., \theta_k)$ with $\min_i(\theta_i - \theta_{i-1}) > 1$:

$$||u||_\infty \le 2|u|_\theta.$$

We now prove a property of the norm $|\cdot|_\theta$ which will be useful later. We use the convention $\theta_{i-1} = -\infty$ if $i = 1$ and $\theta_{i+1} = +\infty$ if $i = k$.

Lemma 7 *Let $\theta = (\theta_1, \dots, \theta_k) \in \mathbb{R}^k$ satisfy $\min_i(\theta_{i+1} - \theta_i) > 8$. For all $X \in E$ there exist $i \in \{1, \dots, k\}$ and $Y \in E$ such that supp $Y \subset [-(\theta_i + \theta_{i+1})/2 - 2, -(\theta_i + \theta_{i-1})/2 + 2]$ and*

$$(P_\theta) \qquad ||Y|| \le 1 \text{ and } |X|_\theta \le 5(X, Y).$$

Proof: Let $m_{i,i+1} = -(\theta_i + \theta_{i+1})/2$. By the definition of $|\cdot|_\theta$, there is i such that $|X|_\theta = ||X||_{W^{1,2}(m_{i,i+1}, m_{i,i-1})}$. Now let R be the function defined by $R = 1$ on $[m_{i,i+1}, m_{i,i-1}]$, $R = 0$ outside $(m_{i,i+1} - 2, m_{i,i-1} + 2)$, R is continuous and linear on each component of $[m_{i,i+1} - 2, m_{i,i-1} + 2] \backslash (m_{i,i+1}, m_{i,i-1})$. We have:

$$\begin{aligned}
(X, RX) &= \int_{\mathbb{R}} R|\dot{X}|^2 + R|X|^2 + \dot{R}X\dot{X} \\
&\ge \int_{m_{i,i+1}}^{m_{i-1,i}} |\dot{X}|^2 + |X|^2 - \int_{m_{i,i+1}-2}^{m_{i,i+1}} \frac{1}{2}|\dot{X}||X| - \int_{m_{i-1,i}}^{m_{i-1,i}+2} \frac{1}{2}|\dot{X}||X| \\
&\ge |X|_\theta^2 - \frac{1}{4}|X|_\theta^2 - \frac{1}{4}|X|_\theta^2 = \frac{1}{2}|X|_\theta^2
\end{aligned}$$

(We just use here that $1/2|\dot{X}||X| \le 1/4(|\dot{X}|^2 + |X|^2)$).

By the same way we easily get

$$||RX||^2 \le 5|X|_\theta^2.$$

Now set $Y = (1/||RX||)RX$. We get $(X, Y) \ge (1/2\sqrt{5})|X|_\theta$ and the proof of lemma 7 is complete.□

We are going to state some easy properties of f_0, f_ϵ, G which will be required in the sequel. The proofs of these properties can be found in the appendix.

Lemma 8 *$\forall b > 0$ $\exists C(b) > 0$ such that: $\forall \epsilon$ with $|\epsilon| < 1$, $\forall s, r, v \in E$ with $||s||_\infty, ||r||_\infty \le b$ we have:*

- **(i)** $|f_\epsilon(s+r) - f_\epsilon(s) - f_\epsilon(r)| \le C(b) \int_{\mathbb{R}} |\dot{r}||\dot{s}| + |r||s|$;
- **(ii)** $|(f'_\epsilon(s+r) - f'_\epsilon(r), v)| \le C(b) \int_{\mathbb{R}} |\dot{s}||\dot{v}| + |s||v|$.

Lemma 9 *There is a positive constant C_4 such that for all $L > 8$, $\forall k$, $\forall (\theta_1, ..., \theta_k) \in \mathbb{R}^k$ with $\min_i(\theta_i - \theta_{i-1}) > L$, $\forall v \in E$ such that $|v|_\theta \le 1$ we have:*

- **(i)** $|f'_0(u^L_{\theta_1} + \ldots + u^L_{\theta_k} + v)|_\theta \le C_4$;
- **(ii)** $|G'(u^L_{\theta_1} + \ldots + u^L_{\theta_k} + v)|_\theta \le C_4$;
- **(iii)** $|f''_0(u^L_{\theta_1} + \ldots + u^L_{\theta_k} + v)X|_\theta \le C_4 |X|_\theta, \quad \forall X \in E$;
- **(iv)** $|G''(u^L_{\theta_1} + \ldots + u^L_{\theta_k} + v)X|_\theta \le C_4 |X|_\theta, \quad \forall X \in E$.

Moreover there is a function r satisfying $\lim_{s\to 0} r(s) = 0$ such that $\forall L > 8$, $\forall k$, $\forall (\theta_1, ..., \theta_k) \in \mathbb{R}^k$ with $\min_i \theta_{i+1} - \theta_i > L$, $\forall v \in E$

- **(v)** $|(f''_0(u^L_{\theta_1} + \ldots + u^L_{\theta_k} + v) - f''_0(u^L_{\theta_1} + \ldots + u^L_{\theta_k}))X|_\theta \le r(|v|_\theta)|X|_\theta$.

Lemma 10

- **(i)** *There is a positive constant C_6 such that*

$$||u_0 - u^L_0|| + ||\dot{u}_0 - \dot{u}^L_0|| + ||\ddot{u}_0 - \ddot{u}^L_0|| = O(\exp(-C_6 L));$$

- **(ii)** *There is a positive constant C_7 such that for all $L > 8$, for all k, for all $(\theta_1, \ldots, \theta_k)$ with $\min_i(\theta_{i+1} - \theta_i) > L$ one has:*

$$|f'_0(u^L_{\theta_1} + \ldots + u^L_{\theta_k})|_\theta = O(\exp(-C_7 L));$$

- **(iii)** *For all $L > 8$, for all $\theta \in \mathbb{R}^k$ such that $\min_i(\theta_{i+1} - \theta_i) > L$, for all $v \in E$ with $|v|_\theta \le 1$*

$$|f''_0(u^L_{\theta_1} + \ldots + u^L_{\theta_k} + v)(\lambda_1 \dot{u}^L_{\theta_1} + \ldots + \lambda_k \dot{u}^L_{\theta_k})|_\theta = O(\max_i |\lambda_i|).$$

Moreover if D^2V is locally Lipschitz continuous then:

$$|f''_0(u^L_{\theta_1} + \ldots + u^L_{\theta_k} + v)(\lambda_1 \dot{u}^L_{\theta_1} + \ldots + \lambda_k \dot{u}^L_{\theta_k})|_\theta = \max_i |\lambda_i| O(|v|_\theta + \exp(-C_8 L))$$

where C_8 is some positive constant.

3.1 The natural constraint $Z^k_{L,\epsilon}$

First we prove a version of lemma 1 in which δ_0 is independent of k. For $\delta > 0$ let:

$$V^{\delta,k}_L = \{(v, \theta) \in E \times \mathbb{R}^k \mid \min_i(\theta_{i+1} - \theta_i) > L, v \in \langle \dot{u}^L_{\theta_1}, \ldots, \dot{u}^L_{\theta_k} \rangle^\perp \ , \ |v|_\theta < \delta\}.$$

Let $h^L : V^{\delta,k}_L \to E$ be defined by $h^L(v, \theta_1, \ldots, \theta_k) = u^L_{\theta_1} + \ldots + u^L_{\theta_k} + v$.

We shall denote by $|\cdot|_\theta$ also the norm on $E \times \mathbb{R}^k$ defined by:

$$|(X, \mu_1, \ldots, \mu_k)|_\theta = \max\{|X|_\theta, |\mu_1|, \ldots, |\mu_k|\}.$$

Lemma 11 *There is $\delta_0 > 0$ such that, if $L > 8$ then $\forall k \in I\!N$, h^L is a diffeomorphism from $V_L^{\delta_0,k}$ onto a neighborhood of Z_L^k.*

Proof:
It goes along the lines of the proof of lemma 1. We use here the new norm $|\cdot|_\theta$ in order to have constants independent of k in the estimates.
Let $U_L^{\delta,k} = \{(v,\theta_1,\ldots,\theta_k) \in E \times \mathbb{R}^k \mid \min_i(\theta_{i+1}-\theta_i) > L\ ,\ |v|_\theta < \delta\}$.
Let $\Psi^L : U_L^{\delta,k} \to E \times \mathbb{R}^k$ be defined by:

$$\Psi^L(\theta, v) = (u_{\theta_1}^L + \ldots + u_{\theta_k}^L + v, (\dot{u}_{\theta_1}^L, v), \ldots, (\dot{u}_{\theta_k}^L, v)).$$

(i) We first prove that there is $\delta_0 > 0$ such that if $L > 8$ then for all k, Ψ^L is a local diffeomorphism on $U_L^{\delta_0,k}$. Indeed we have:

$$d_{(v,\theta)}\Psi^L(X,\lambda_1,\ldots,\lambda_k) = (\lambda_1\dot{u}_{\theta_1}^L + \ldots + \lambda_k\dot{u}_{\theta_k}^L + X, (\dot{u}_{\theta_1}^L, X) + \lambda_1(\ddot{u}_{\theta_1}^L, v), \ldots, (\dot{u}_{\theta_k}^L, X) + \lambda_k(\ddot{u}_{\theta_k}^L, v)).$$

Let B denote the linear operator defined on $E \times \mathbb{R}^k$ by:

$$B(X,\lambda_1,\ldots,\lambda_k) = (X + \lambda_1\dot{u}_{\theta_1}^L + \ldots + \lambda_k\dot{u}_{\theta_k}^L, (X, \dot{u}_{\theta_1}^L), \ldots, (X, \dot{u}_{\theta_k}^L)).$$

Since

$$\text{supp } \ddot{u}_{\theta_i}^L \subset I_i = [-(\theta_{i+1}+\theta_i)/2, -(\theta_i+\theta_{i-1})/2] \tag{3.1}$$

we have: $|(\ddot{u}_{\theta_i}^L, v)| \leq ||\ddot{u}_{\theta_i}^L|| \cdot |v|_\theta$. Hence as in the proof of lemma 1:

$$|(d_{(v,\theta)}\psi^L - B)(X,\lambda_1,\ldots,\lambda_k)|_\theta \leq \delta_0 C|(X,\lambda_1,\ldots,\lambda_k)|_\theta. \tag{3.2}$$

For all $X \in E$ we can write $X = Y + \mu_1\dot{u}_{\theta_1}^L + \ldots + \mu_k\dot{u}_{\theta_k}^L$ where $Y \in \langle \dot{u}_{\theta_1}^L, \ldots, \dot{u}_{\theta_k}^L \rangle^\perp$. Thus

$$|B(X,\lambda_1,\ldots,\lambda_k)|_\theta = \max(|Y + (\mu_1+\lambda_1)\dot{u}_{\theta_1}^L + \ldots + (\mu_k+\lambda_k)\dot{u}_{\theta_k}^L|_\theta, ||\dot{u}_0^L||^2|\mu_1|, \ldots, ||\dot{u}_0^L||^2|\mu_k|).$$

Now we have

$$|Y + (\mu_1+\lambda_1)\dot{u}_{\theta_1}^L + \ldots + (\mu_k+\lambda_k)\dot{u}_{\theta_k}^L|_\theta = \max_i(||Y + (\mu_i+\lambda_i)\dot{u}_{\theta_i}^L||_{W^{1,2}(I_i)}).$$

Moreover, since Y and $\dot{u}_{\theta_i}^L$ are orthogonal in $W^{1,2}(I_i)$:

$$||Y + (\mu_i+\lambda_i)\dot{u}_{\theta_i}^L||_{W^{1,2}(I_i)} \geq \frac{1}{2}(||Y||_{W^{1,2}(I_i)} + |\mu_i+\lambda_i|||\dot{u}_0^L||).$$

Hence

$$|Y + (\mu_1+\lambda_1)\dot{u}_{\theta_1}^L + \ldots + (\mu_k+\lambda_k)\dot{u}_{\theta_k}^L|_\theta \geq C'(|Y|_\theta + \max_i|\mu_i+\lambda_i|)$$

and as in the proof of lemma 1 we get:

$$|B(X,\lambda_1,\ldots,\lambda_k)|_\theta \geq C''|(X,\lambda_1,\ldots,\lambda_k)|_\theta. \tag{3.3}$$

Choosing $\delta_0 > 0$ such that $\delta_0 C < C''/2$ we get by (3.2) and (3.3) that for all k, for all $L > 8$, for all $(v,\theta_1,\ldots,\theta_k) \in U_L^{\delta_0,k}$

$$|d_{(v,\theta_1,\ldots,\theta_k)}\psi^L(X,\lambda_1,\ldots,\lambda_k)|_\theta \geq \frac{C''}{2}|(X,\lambda_1,\ldots,\lambda_k)|_\theta. \tag{3.4}$$

Since $d_{(v,\theta_1,\ldots,\theta_k)}\psi^L$ has the form "Id + compact" we derive from (3.4) that it is an isomorphism and ψ^L is a local diffeomorphism on $U_L^{\delta_0,k}$.

(ii) We must justify that, provided δ_0 is small enough, ψ^L is injective on $U_L^{\delta_0,k}$. We have that for all $\gamma > 0$ there is $\eta > 0$ independent of $k, L > 8$, such that for all $(v, \theta_1, \ldots, \theta_k)$, $(v', \theta'_1, \ldots, \theta'_k)$ in $U_L^{\delta_0,k}$ $\max(|v - v'|_\theta, |\theta_1 - \theta'_1|, \ldots, |\theta_k - \theta'_k|) < \eta$ implies

$$|(d_{(v,\theta_1,\ldots,\theta_k)}\psi^L - d_{(v',\theta'_1,\ldots,\theta'_k)}\psi^L)(X, \lambda_1, \ldots, \lambda_k)|_\theta < \gamma|(X, \lambda_1, \ldots, \lambda_k)|_\theta.$$

This property can be easily checked using the fact that for η small enough and $L > 8$, $|\theta_i - \theta'_i| < \eta$ implies $\text{supp}\, u^L_{\theta_i} \cup \text{supp}\, u^L_{\theta'_i} \subset I_i$ and that $\dot{u}^L_{\theta_i} - \dot{u}^L_{\theta'_i} \to 0$, $\ddot{u}^L_{\theta_i} - \ddot{u}^L_{\theta'_i} \to 0$ as $|\theta_i - \theta'_i| \to 0$ independently of $L > 8$.

This uniform continuity property combined with (3.4) yields the existence of $\nu > 0$ (independent of L, k) such that:

$$0 < \max(|v - v'|_\theta, |\theta_1 - \theta'_1|, \ldots, |\theta_k - \theta'_k|) < \nu \Rightarrow \psi^L(v, \theta_1, \ldots, \theta_k) \neq \psi^L(v, \theta'_1, \ldots, \theta'_k).$$

The proof of injectivity (provided δ_0 is small enough) now presents no difference with that given in lemma 1, so we omit it. $\square$.

As in section 2 we consider the following function:

$$H^L \ : \ \mathbb{R} \times \mathbb{R}^k \times E \times \mathbb{R}^k \to E \times \mathbb{R}^k$$

with components $H_1^L \in E$ and $H_2^L \in \mathbb{R}^k$ given by:

$$\begin{aligned} H_1^L(\epsilon, \theta_1, \ldots, \theta_k, v, \alpha_1, \ldots, \alpha_k) &= f'_\epsilon(u^L_{\theta_1} + \ldots + u^L_{\theta_k} + v) - \sum\nolimits_{i=1}^{i=k} \alpha_i \dot{u}^L_{\theta_i}, \\ H_2^L(\epsilon, \theta_1, \ldots, \theta_k, v, \alpha_1, \ldots, \alpha_k) &= ((v, \dot{u}^L_{\theta_1}), \ldots, (v, \dot{u}^L_{\theta_k})). \end{aligned}$$

We prove here a modified version of lemma 2 in which the constants can be taken independent of k.

Lemma 12 *There exist positive constants C_9, L_1 such that for all $L > L_1$,$\forall k$, for all $\theta = (\theta_1, \ldots, \theta_k)$ with $\min_i(\theta_i - \theta_{i-1}) > L$ and for all $(X, \mu_1, \ldots, \mu_k)$:*

$$\left| \frac{\partial H^L}{\partial(v, \alpha)}_{|(0,\theta,0)} (X, \mu_1, \ldots, \mu_k) \right|_\theta \geq C_9 |(X, \mu_1, \ldots, \mu_k)|_\theta. \tag{3.5}$$

Proof:

Arguing by contradiction we assume that the statement in lemma 12 does not hold.

Then we can define sequences L_n, k_n, $\theta^n = (\theta^n_1, \ldots, \theta^n_{k_n})$, $(X_n, \mu^n_1, \ldots, \mu^n_{k_n}) \in E \times \mathbb{R}^{k_n}$ such that $L_n \to +\infty$, $\min_i(\theta^n_{i+1} - \theta^n_i) > L_n$, $|(X_n, \mu^n_1, \ldots, \mu^n_{k_n})|_{\theta^n} = 1$, and:

$$\left| \frac{\partial H^{L_n}}{\partial(v, \alpha)}_{|(0,\theta^n,0)} (X_n, \mu^n_1, \ldots, \mu^n_{k_n}) \right|_{\theta^n} \to 0, \tag{3.6}$$

which means:

$$|f_0{}''(u^{L_n}_{\theta^n_1} + \ldots + u^{L_n}_{\theta^n_{k_n}})X_n - \sum_{i=1}^{i=k_n} \mu^n_i \dot{u}^{L_n}_{\theta^n_i}|_{\theta^n} \to 0 \tag{3.7}$$

and

$$\max_{i=1,\ldots,k_n}|(X_n, \dot{u}^{L_n}_{\theta^n_i})| \to 0. \tag{3.8}$$

By the definition of $|(X_n, \mu^n_1, \ldots, \mu^n_{k_n})|_{\theta^n}$ there is $i_n \in \{1, \ldots, k_n\}$ such that

$$1 \leq ||X_n||_{W^{1,2}(I^n_{i_n})} + |\mu^n_{i_n}| \leq 2 \tag{3.9}$$

where $I^n_{i_n} = (-(\theta^n_{i_n} + \theta^n_{i_n+1})/2, -(\theta^n_{i_n} + \theta^n_{i_n-1})/2)$.

Using the invariance of f_0 and of the scalar product $(\cdot,\cdot)$ under the action of $\mathbb{R}$ we can assume that $\theta^n_{i_n} = 0$ for any n. Hence $[-L_n/2, L_n/2] \subset I^n_{i_n}$. Define a sequence R_n of cut-off functions such that $R_n(\mathbb{R}) \subset [0,1]$, $R_n = 1$ on $I^n_{i_n}$, $R_n = 0$ outside $I^n_{i_n} + [-2,2]$ and R_n is continuous and linear on each component of $(I^n_{i_n} + [-2,2]) \backslash I^n_{i_n}$. Set $Y_n = R_n X_n$.

As in the proof of lemma 7 we have

$$||Y_n||^2 \leq 5|X_n|^2_{\theta^n} \leq 5.$$

Hence up to a subsequence $Y_n \rightharpoonup X \in E$. Note that $X_n = Y_n$ on $[-L_n/2, L_n/2]$ and $\lim_{n\to+\infty} L_n = +\infty$. As a consequence $X_n \to X$ in $L^2_{loc}(\mathbb{R})$.

Since $|\mu^n_{i_n}|$ is bounded we can also assume that $\mu^n_{i_n} \to \mu \in \mathbb{R}$. We are going to show that $X = 0$ and $\mu = 0$.

Let $g \in E$ be fixed and have its support in a compact interval J. We have from (3.7) that:

$$(f_0''(u^{L_n}_{\theta^n_1} + \ldots + u^{L_n}_{\theta^n_{k_n}})X_n, g) - \sum_{i=1}^{i=k_n} \mu^n_i(\dot{u}^{L_n}_{\theta^n_i}, g) \to 0. \tag{3.10}$$

Now for n large enough supp $g \subset I^n_{i_n}$, which implies that supp $g \cap$ supp $u^{L_n}_{\theta^n_i} = \emptyset$ for $i \neq i_n$.

Hence, since $\theta_{i_n} = 0$, (3.10) is equivalent to:

$$(X_n, g) - \int_{\mathbb{R}} D^2V(u^{L_n}_0)X_n g - \mu^n_{i_n}(\dot{u}^{L_n}_0, g) \to 0. \tag{3.11}$$

We have $u^{L_n}_0 \to u_0$ in L^∞. Hence, by the uniform continuity of D^2V on bounded subsets of $\mathbb{R}^n$, $D^2V(u^{L_n}_0) \to D^2V(u_0)$ in L^∞. Therefore since $X_n \to X$ in $L^2_{loc}(\mathbb{R})$ and g is compactly supported,

$$\int_{\mathbb{R}} D^2V(u^{L_n}_0)X_n g \to \int_{\mathbb{R}} D^2V(u_0)Xg.$$

Furthermore $\mu^n_1(\dot{u}^{L_n}_0, g) \to \mu(\dot{u}_0, g)$ and for n large enough $(X_n, g) = (Y_n, g)$. So $(X_n, g) \to (X, g)$ and we can derive from (3.11):

$$(X, g) + \int_{\mathbb{R}} D^2V(u_0)Xg - \mu(\dot{u}_0, g) = 0. \tag{3.12}$$

Since this equality holds for all $g \in E$ with compact support we get:

$$f_0''(u_0)X = \mu\dot{u}_0. \tag{3.13}$$

Since f_0'' is symmetric and $\dot{u}_0 \in \ker f_0''(u_0)$ (3.13) implies that $\mu = 0$ and $X \in \ker f_0''(u_0) = TZ_{u_0} = \mathbb{R}\dot{u}_0$. Now from (3.8):

$$(Y_n, \dot{u}^{L_n}_0) = (X_n, \dot{u}^{L_n}_0) \to 0.$$

Hence $(X, \dot{u}_0) = 0$ Finally we get $X = 0$.

To get a contradiction we write that from (3.7)

$$(f_0''(u_0^{L_n}+\ldots+u_{\theta_{k_n}^n}^{L_n})X_n,Y_n)-\sum_{i=1}^{i=k_n}\mu_{i_n}^n(\dot{u}_{\theta_i^n}^{L_n},Y_n)\to 0$$

which implies, since supp $Y_n\cap$ supp $u_{\theta_i^n}^{L_n}=\emptyset$ for $i\neq i_n$ and $(Y_n,\dot{u}_0^{L_n})\to 0$:

$$(X_n,Y_n)-\int_{\mathbb{R}}D^2V(u_0^{L_n})X_nY_n\to 0. \tag{3.14}$$

Now $X_nY_n=R_nX_n^2$ is bounded in L^1 norm and tends to 0 in L^1_{loc}. Moreover using that $D^2V(0)=0$, $|u_0^{L_n}(t)|\leq|u_0(t)|$ and $|u_0(t)|\to 0$ as $|t|\to+\infty$ we can write

$$||D^2V(u_0^{L_n})||_{L^\infty[-A,A]^c}\leq C_A$$

with $\lim_{A\to\infty}C_A=0$.

Hence:

$$\int_{\mathbb{R}}D^2V(u_0^{L_n})X_nY_n\to 0.$$

This latter limit and (3.14) implies that $(X_n,Y_n)\to 0$.

Now

$$\begin{aligned}(X_n,Y_n) &= \int_{I_{i_n}^n+[-2,2]}R_n|\dot{X}_n|^2+R_n|X_n|^2+\dot{R}_n\dot{X}_nX_n\\ &\geq ||X_n||^2_{W^{1,2}(I_{i_n}^n)}-\frac{1}{4}||X_n||^2_{W^{1,2}(I_{i_n-1}^n)}-\frac{1}{4}||X_n||^2_{W^{1,2}(I_{i_n+1}^n)}\\ &\geq ||X_n||^2_{W^{1,2}(I_{i_n}^n)}-\frac{1}{2}|X_n|^2_{\theta^n}\geq||X_n||^2_{W^{1,2}(I_{i_n}^n)}-\frac{1}{2}.\end{aligned}$$

This contradicts $(X_n,Y_n)\to 0$ because of (3.9) and the fact that $\mu_{i_n}^n\to 0$. □

Remark 7 *By lemma 9-(iv)-(v) it is clear that lemma 12 implies the following properties of H^L: Provided δ_0 and ϵ_0 are small enough we have:*

- **(i)** *For all $L>L_1$, for all $\epsilon\in(-\epsilon_0,\epsilon_0)$, for all $\theta\in\mathbb{R}^k$ such that $\min_i(\theta_{i+1}-\theta_i)>L$, for all $v\in E$ with $|v|_\theta<\delta_0$, $\dfrac{\partial H^L}{\partial(v,\alpha)}_{|(\epsilon,\theta,v)}$ is an isomorphism and:*

$$\left|\left[\frac{\partial H^L}{\partial(v,\alpha)}_{|(\epsilon,\theta,v)}\right]^{-1}(X,\mu_1,\ldots,\mu_k)\right|_\theta\leq\frac{2}{C_9}|(X,\mu_1,\ldots,\mu_k)|_\theta.$$

- **(ii)** *For all $L>L_1$, for all $\epsilon\in(-\epsilon_0,\epsilon_0)$, for all $\theta\in\mathbb{R}^k$ such that $\min_i(\theta_{i+1}-\theta_i)>L$, for all $v,v'\in E$ such that $|v|_\theta,|v'|_\theta<\delta_0$, for all $\alpha,\alpha'\in\mathbb{R}^k$*

$$|H^L(\epsilon,\theta,v,\alpha)-H^L(\epsilon,\theta,v',\alpha')|_\theta\geq\frac{C_9}{2}|(v-v',\alpha-\alpha')|_\theta.$$

Remark 8 *It can be readily checked that if a sequence w_n with $w_n \in E$ is bounded in L^∞ and converges in $W^{1,2}_{loc}$ then $f'_\epsilon(w_n)$ converges in $W^{1,2}_{loc}$ and is bounded in L^∞. Hence it makes sense to define $f'_\epsilon(u) \in W^{1,2}_{loc} \cap L^\infty$ for $u \in W^{1,2}_{loc} \cap L^\infty$.*

Let $(\dots, \theta_{-k}, \dots, \theta_0, \dots, \theta_k, \dots)$ be an infinite sequence such that $\theta_{i+1} - \theta_i > L$. We can still define the norm $|\cdot|_\theta$. Note that $\sum_{i=-\infty}^{i=+\infty} u^L_{\theta_i} \in W^{1,2}_{loc} \cap L^\infty$ and that $|\sum_{i=-\infty}^{i=+\infty} u^L_{\theta_i}|_\theta = ||u^L_0||$.

It is easy to see using property (P_θ) of lemma 7 that if $|v|_\theta < +\infty$ then $|f'_\epsilon(\sum_{i=-\infty}^{i=+\infty} u^L_{\theta_i} + v)|_\theta < +\infty$. Now since in remark 7-(ii) the estimate is independent of k it clearly implies that if $v, v' \in W^{1,2}_{loc}$ satisfies $|v|_\theta, |v'|_\theta < \delta_0$ and $\forall i \in \mathbb{Z}$, $(v, \dot{u}^L_{\theta_i}) = (v', \dot{u}^L_{\theta_i}) = 0$ then

$$|f'_\epsilon(\sum_{i\in\mathbb{Z}} u^L_{\theta_i} + v) - \sum_{i\in\mathbb{Z}} \alpha_i \dot{u}^L_{\theta_i} - f'_\epsilon(\sum_{i\in\mathbb{Z}} u^L_{\theta_i} + v') + \sum_{i\in\mathbb{Z}} \beta_i \dot{u}^L_{\theta_i}|_\theta \geq \frac{C_9}{2}(|v - v'|_\theta + \sup_i |\alpha_i - \beta_i|).$$

Lemma 3 of section 2 can be modified in such a way that L_2 does not depend on k. We have:

Lemma 13 *There exists $L_2 \geq L_1$ such that for all $L > L_2$, for all k there exist C^1 functions $w(L,\cdot)$ and $\alpha_i(L,\cdot)$ defined on $\{\theta \in \mathbb{R}^k \mid \min_i(\theta_{i+1} - \theta_i) > L\}$ such that $w(L, \theta_1, \dots, \theta_k) \in \langle \dot{u}^L_{\theta_1}, \dots, \dot{u}^L_{\theta_k}\rangle^\perp$, $|w(L, \theta_1, \dots, \theta_k)|_\theta \leq \delta_0$ and*

$$f'_0(u^L_{\theta_1} + \ldots + u^L_{\theta_k} + w(L, \theta_1, \dots, \theta_k)) = \sum_{i=1}^k \alpha_i(L, \theta) \dot{u}^L_{\theta_i}. \tag{3.15}$$

Moreover there are constants C_{10} and C_{11} such that $|w(L, \theta_1, \dots, \theta_k)|_\theta = O(\exp(-C_{10}L))$ and $\max_i |\alpha_i(L, \theta)| = O(\exp(-C_{11}L))$.

Proof:

The existence proof is exactly the same as for lemma 3. The only change is that we use for each $\theta = (\theta_1, \dots, \theta_k)$ the norm $|\cdot|_\theta$ to get the existence of $w(L, \theta_1, \dots, \theta_k)$.

The C^1 regularity of w and of α_i is as for lemma 3 a consequence of the implicit function theorem since for any fixed k the $|\cdot|_\theta$ norm is equivalent to the $||\cdot||$ norm.

So it remains to justify the estimates on $|w|_\theta$ and $|\alpha_i|$.

The functions $w, \alpha = (\alpha_1, \dots, \alpha_k)$ are defined by the equation $H^L(0, \theta, w, \alpha) = 0$. Therefore, by remark 7,

$$|-H^L(0, \theta, 0, 0)|_\theta = |H^L(0, \theta, w, \alpha) - H^L(0, \theta, 0, 0)|_\theta \geq \frac{C_9}{2}|(w, \alpha_1, \dots, \alpha_k)|_\theta.$$

Hence

$$|(w, \alpha_1, \dots, \alpha_k)|_\theta \leq \frac{2}{C_9}|H^L(0, \theta, 0, 0)|_\theta.$$

Now by lemma 10

$$|H^L(0, \theta, 0, 0)|_\theta = |f'_0(u^L_{\theta_1} + \cdots + u^L_{\theta_k})|_\theta = O(\exp(-C_7 L))$$

and we get the estimates. □

Now we can state another lemma analogous to lemma 4.

Lemma 14 *There exists $\epsilon_1 > 0$ such that for every k, for every $L > L_2$ there exist C^1 functions $\bar{w}(L,\cdot)$ and $\tilde{\alpha}_i(L,\cdot)$ $(1 \leq i \leq k)$ defined on $(-\epsilon_1, \epsilon_1) \times \{(\theta_1, \dots, \theta_k) \mid \min_i(\theta_{i+1} - \theta_i) > L\}$ such that:*

- $\bar{w}(L,0,\theta_1,\ldots,\theta_k)=0$;
- $f'_\epsilon(u^L_{\theta_1}+\ldots+u^L_{\theta_k}+w(L,\theta_1,\ldots,\theta_k)+\bar{w}(L,\epsilon,\theta_1,\ldots,\theta_k))=\sum_{i=1}^k\tilde{\alpha}_i(L,\epsilon,\theta)\dot{u}^L_{\theta_i}$;
- $\bar{w}(L,\epsilon,\theta_1,\ldots,\theta_k)\in\langle\dot{u}^L_{\theta_1},\ldots,\dot{u}^L_{\theta_k}\rangle^\perp$, $|\bar{w}(L,\epsilon,\theta_1,\ldots,\theta_k)|_\theta<\delta_0$.

Moreover there exist positive constants C_{12} and C_{13} such that, for all L, for all k, for all $(\theta_1,\ldots,\theta_k)$ such that $\min_i(\theta_{i+1}-\theta_i)>L$, $|\bar{w}(L,\epsilon,\theta_1,\ldots,\theta_k)|_\theta\le C_{12}|\epsilon|$, $\max_i|\tilde{\alpha}_i(L,\epsilon,\theta)|\le C_{12}|\epsilon|+O(\exp(-C_{13}L))$.

We will use also the notations $\bar{w}_\epsilon(L,\theta_1,\ldots,\theta_k)=\bar{w}(L,\epsilon,\theta_1,\ldots,\theta_k)$ and $\tilde{w}_\epsilon(L,\theta)=w(L,\theta)+\bar{w}_\epsilon(L,\theta)$.

Proof: As for lemma 13, the existence and C^1 regularity proof can be easily carried out by the contraction mapping theorem and the implicit function theorem, using remark 7 and lemma 9-(iv)-(v). We will only justify the estimates on $|\bar{w}_\epsilon(L,\theta)|_\theta$ and on $|\tilde{\alpha}_i(L,\epsilon,\theta)|$.
In fact by remark 7

$$|H^L(0,\theta,\tilde{w}_\epsilon(L,\theta),\tilde{\alpha}(L,\epsilon,\theta))-H^L(0,\theta,w(L,\theta),\alpha(L,\theta))|_\theta\ge\frac{C_9}{2}|(\bar{w}(L,\epsilon,\theta);\tilde{\alpha}(L,\epsilon,\theta)-\alpha(L,\theta))|_\theta.$$

Now

$$H^L(0,\theta,w(L,\theta),\alpha(L,\theta))=0$$

and

$$|H^L(0,\theta,\tilde{w}(L,\epsilon,\theta),\tilde{\alpha}(L,\epsilon,\theta))|_\theta=|\epsilon G'(u^L_{\theta_1}+\ldots+u^L_{\theta_k}+\tilde{w}(L,\epsilon,\theta))|_\theta.$$

The desired estimates follow by lemma 9-(ii).□

Finally, for $L>L_2$ and $|\epsilon|<\epsilon_1$, we define:

$$Z^k_{L,\epsilon}=\{u^L_{\theta_1}+\ldots+u^L_{\theta_k}+w(L,\theta_1,\ldots,\theta_k)+\bar{w}(L,\epsilon,\theta_1,\ldots,\theta_k)\mid\min_i(\theta_{i+1}-\theta_i)>L\}.$$

By lemma 11 $Z^k_{L,\epsilon}$ is a C^1 k-dimensional submanifold of E diffeomorphic to Z^k_L.
Now it is easy to show that:

Lemma 15 *$Z^k_{L,\epsilon}$ is a natural constraint for f_ϵ.*

We finish this subsection with a lemma providing an estimate on $\partial_\theta\tilde{w}_\epsilon$ which will be used later.

Lemma 16 *For $L>L_2$ and $\theta\in I\!R^k$ such that $\min_i(\theta_{i+1}-\theta_i)>L$, for $\epsilon\in(-\epsilon_1,\epsilon_1)$, for all $(\lambda_1,\ldots,\lambda_k)\in I\!R^k$:*

$$\left|\lambda_1\frac{\partial\tilde{w}_\epsilon}{\partial\theta_1}(L,\theta)+\ldots+\lambda_k\frac{\partial\tilde{w}_\epsilon}{\partial\theta_k}(L,\theta)\right|_\theta=O(\max_i|\lambda_i|).$$

Moreover if D^2V is locally Lipschitz continuous then we have the sharper estimate:

$$\left|\lambda_1\frac{\partial\tilde{w}_\epsilon}{\partial\theta_1}+\ldots+\lambda_k\frac{\partial\tilde{w}_\epsilon}{\partial\theta_k}\right|_\theta=O(\epsilon+\exp(-C_{14}L))\max_i|\lambda_i|$$

for some positive constant C_{14}.

Proof: By the implicit function theorem we have:

$$(\frac{\partial \tilde{w}_\epsilon}{\partial \theta_i}, \frac{\partial \tilde{\alpha}_\epsilon}{\partial \theta_i}) = -[\frac{\partial H^L}{\partial(v,\alpha)}]^{-1} \frac{\partial H^L}{\partial \theta_i}(\epsilon, \theta, \tilde{w}_\epsilon, \tilde{\alpha}_\epsilon).$$

Hence using remark 7 we can obtain:

$$\left| \lambda_1 \frac{\partial \tilde{w}_\epsilon}{\partial \theta_1} + \ldots + \lambda_k \frac{\partial \tilde{w}_\epsilon}{\partial \theta_k} \right|_\theta \leq \frac{2}{C_9} \left| \sum_{i=1}^k \lambda_i \frac{\partial H^L}{\partial \theta_i}(\epsilon, \theta, \tilde{w}_\epsilon, \tilde{\alpha}_\epsilon) \right|_\theta . \tag{3.16}$$

Now

$$\sum_{i=1}^k \lambda_i \frac{\partial H_1^L}{\partial \theta_i}(\epsilon, \theta, \tilde{w}_\epsilon, \tilde{\alpha}_\epsilon) = f_0''(u_{\theta_1}^L + \ldots + u_{\theta_k}^L + \tilde{w}_\epsilon)(\sum_{i=1}^k \lambda_i \dot{u}_{\theta_i}^L) - \sum_{i=1}^k \lambda_i \tilde{\alpha}_{\epsilon,i} \ddot{u}_{\theta_i}^L.$$

Hence by lemma 10-(iii) and the estimate on $\max_i |\tilde{\alpha}_{i,\epsilon}|$ we can write

$$\left| \sum_{i=1}^k \lambda_i \frac{\partial H_1^L}{\partial \theta_i}(\epsilon, \theta, \tilde{w}_\epsilon(L,\theta), \tilde{\alpha}_\epsilon(L,\theta)) \right|_\theta = O(\max_i |\lambda_i|).$$

Moreover, if $D^2 V$ is locally Lipschitz continuous, then, by the estimate on $|\tilde{w}_\epsilon|_\theta$, we have:

$$\left| \sum_{i=1}^k \lambda_i \frac{\partial H_1^L}{\partial \theta_i}(\epsilon, \theta, \tilde{w}_\epsilon(L,\theta), \tilde{\alpha}_\epsilon(L,\theta)) \right|_\theta = \max_i |\lambda_i| O(\epsilon + \exp(-CL))$$

for some positive constant C. Furthermore

$$\sum_{i=1}^k \lambda_i \frac{\partial H_2^L}{\partial \theta_i} = (\lambda_1 (\ddot{u}_{\theta_1}^L, \tilde{w}_\epsilon), \ldots, \lambda_k (\ddot{u}_{\theta_k}^L, \tilde{w}_\epsilon))$$

and $|(\ddot{u}_{\theta_i}^L, \tilde{w}_\epsilon)| \leq ||\ddot{u}_{\theta_i}^L|| \cdot |\tilde{w}_\epsilon|_\theta = O(\epsilon + \exp(-C'L))$ by lemmas 13 and 14. Finally we get

$$\left| \sum_{i=1}^k \lambda_i \frac{\partial H^L}{\partial \theta_i}(\epsilon, \theta, \tilde{w}_\epsilon, \tilde{\alpha}_\epsilon) \right|_\theta = O(\max_i |\lambda_i|)$$

and if $D^2 V$ is locally Lipschitz continuous then

$$\left| \sum_{i=1}^k \lambda_i \frac{\partial H^L}{\partial \theta_i}(\epsilon, \theta, \tilde{w}_\epsilon, \tilde{\alpha}_\epsilon) \right|_\theta = O(\epsilon + \exp(-C''L)) \max_i |\lambda_i|.$$

From (3.16) we get the desired estimates.

3.2 Expression of f_ϵ on $Z_{L,\epsilon}^k$

By lemma 15 we are led, in order to find k-bump solutions and then solutions with an infinite number of bumps, to look for the critical points of the functional f_ϵ restricted to the k-dimensional manifold $Z_{L,\epsilon}^k$.

We will need the following lemma, whose proof is given in the appendix:

Lemma 17 *There is a positive constant* C_{15} *such that for all* $\theta = (\theta_1, \ldots, \theta_k) \in I\!R^k$ *with* $\min_i(\theta_{i+1} - \theta_i) > L$*:*

$$\forall i \in \{1, \ldots, k-1\} \quad ||\tilde{w}_\epsilon(L, \theta)||_{W^{1,2}(J_i)} = O(\exp(-C_{15}(\theta_{i+1} - \theta_i)))$$

where $J_i = (-(\theta_i + \theta_{i+1})/2 - 2, -(\theta_i + \theta_{i+1})/2 + 2)$.

In the next lemma we find a suitable expression for the functional f_ϵ restricted to $Z^k_{L,\epsilon}$.

Lemma 18 *For all* $|\epsilon| < \epsilon_1$*, for all* $L > L_2$*, for all* k*, for all* $(\theta_1, \ldots, \theta_k) \in I\!R^k$ *with* $\min_i(\theta_{i+1} - \theta_i) > L$ *there results:*

$$f_\epsilon(u^L_{\theta_1} + \ldots + u^L_{\theta_k} + w(L, \theta) + \bar{w}_\epsilon(L, \theta)) = kb + \epsilon(G(u_{\theta_1}) + \ldots + G(u_{\theta_k})) + \beta(L, \epsilon, \theta) \tag{3.17}$$

where β *has the following property: there is a positive constant* C_{16} *such that, if* θ'_i *satisfies* $\theta'_i - \theta_{i-1} > L$ *and* $\theta_{i+1} - \theta'_i > L$ *then*

$$|\beta(L, \epsilon, \theta_1, \ldots, \theta_{i-1}, \theta'_i, \theta_{i+1}, \ldots, \theta_k) - \beta(L, \epsilon, \theta_1, \ldots, \theta_{i-1}, \theta_i, \theta_{i+1}, \ldots, \theta_k)|$$
$$= O(\exp(-C_{16}L)) + \epsilon o_{\epsilon,L}(1). \tag{3.18}$$

Proof:
For the sake of simplicity we shall give the proof of (3.18) for $i = 1$ only.
Let $\phi \in C^\infty(I\!R, [0,1])$ satisfy: $\phi = 0$ on $(-\infty, -1]$, $\phi = 1$ on $[1, +\infty)$, $|\dot{\phi}| \le 1$ on $I\!R$.
We shall use the abbreviations: $S = u^L_{\theta_2} + \ldots + u^L_{\theta_k}$ and $\phi_1(\cdot) = \phi(\frac{\theta_1 + \theta_2}{2} + \cdot)$.
We first prove:

$$\begin{aligned} \beta(L, \epsilon, \theta_1, \ldots, \theta_k) &= f_\epsilon(u^L_{\theta_1} + \phi_1(w + \bar{w}_\epsilon)) + f_\epsilon(S + (1 - \phi_1)(w + \bar{w}_\epsilon)) \\ &- \epsilon(G(u_{\theta_1}) + \ldots + G(u_{\theta_k})) - kb + O(\exp(-CL)). \end{aligned} \tag{3.19}$$

Indeed
$$\beta(L, \epsilon, \theta_1, \ldots, \theta_k) = f_\epsilon(u^L_{\theta_1} + S + w + \bar{w}_\epsilon) - \epsilon(G(u_{\theta_1}) + \ldots + G(u_{\theta_k})) - kb.$$

Moreover we can write

$$f_\epsilon(u^L_{\theta_1} + S + w + \bar{w}_\epsilon) = f_\epsilon(u^L_{\theta_1} + \phi_1(w + \bar{w}_\epsilon)) + f_\epsilon(S + (1 - \phi_1)(w + \bar{w}_\epsilon)) + r(L, \epsilon, \theta_1, \ldots, \theta_k).$$

By lemma 8-(i), using that supp $u^L_{\theta_i} \subset [-\theta_i - L/4, -\theta_i + L/4]$, we have that

$$\begin{aligned} |r(L, \epsilon, \theta_1, \ldots, \theta_k)| &\le C \int_{I\!R} |\frac{d}{dt}(u^L_{\theta_1} + \phi_1(w + \bar{w}_\epsilon))||\frac{d}{dt}(S + (1 - \phi_1)(w + \bar{w}_\epsilon))| \\ &+ C \int_{I\!R} |u^L_{\theta_1} + \phi_1(w + \bar{w}_\epsilon)||S + (1 - \phi_1)(w + \bar{w}_\epsilon)| \\ &\le C \int_{I\!R} |\frac{d}{dt}(\phi_1(w + \bar{w}_\epsilon))||\frac{d}{dt}((1 - \phi_1)(w + \bar{w}_\epsilon))| + |\phi_1(w + \bar{w}_\epsilon)||(1 - \phi_1)(w + \bar{w}_\epsilon)|. \end{aligned}$$

Since (supp $\frac{d}{dt}(\phi_1(w + \bar{w}_\epsilon))$ $\cap$ supp $\frac{d}{dt}((1 - \phi_1)(w + \bar{w}_\epsilon))) \subset [-(\theta_1 + \theta_2)/2 - 1, -(\theta_1 + \theta_2)/2 + 1]$, by lemma 17

$$\int_{I\!R} |\frac{d}{dt}(\phi_1(w + \bar{w}_\epsilon))||\frac{d}{dt}((1 - \phi_1)(w + \bar{w}_\epsilon))| = O(\exp(-C(\theta_2 - \theta_1))) = O(\exp(-CL)).$$

We get the same estimate for $\int_{\rm I\!R} |\phi_1(w+\bar{w}_\epsilon)||(1-\phi_1)(w+\bar{w}_\epsilon)|$.

Hence we derive that $|r(\epsilon,\theta_1,\dots,\theta_k)| = O(\exp(-CL))$ and (3.19) holds.

We now prove:

$$f_\epsilon(u^L_{\theta_1} + \phi_1(w+\bar{w}_\epsilon)) - \epsilon G(u_{\theta_1}) = b + O(\exp(-CL)+\epsilon^2). \tag{3.20}$$

We recall that b is the critical level of f_0 associated to the critical point u_0. Indeed,

$$f_\epsilon(u^L_{\theta_1} + \phi_1(w+\bar{w}_\epsilon)) - \epsilon G(u_{\theta_1}) = f_0(u^L_{\theta_1} + \phi_1(w+\bar{w}_\epsilon)) + \epsilon G(u^L_{\theta_1} + \phi_1(w+\bar{w}_\epsilon)) - \epsilon G(u_{\theta_1}).$$

As in the proof of lemma 6 we can write:

$$f_0(u^L_{\theta_1} + \phi_1(w+\bar{w}_\epsilon)) = f_0(u^L_{\theta_1}) + (f_0'(u^L_{\theta_1}), \phi_1(w+\bar{w}_\epsilon)) + O(||\phi_1(w+\bar{w}_\epsilon)||^2).$$

Since supp $\phi_1 \subset (-(\theta_1+\theta_2)/2+1, +\infty)$ by lemmas 13 and 14 we get:

$$||\phi_1(w+\bar{w}_\epsilon)||^2 = O(|w+\bar{w}_\epsilon|^2_\theta) = O(\epsilon^2 + \exp(-CL)).$$

Moreover by lemma 10-(i)

$$\begin{aligned}(f_0'(u^L_{\theta_1}), \phi_1(w+\bar{w}_\epsilon)) &= (f_0'(u_{\theta_1}), \phi_1(w+\bar{w}_\epsilon)) + O(||u_{\theta_1} - u^L_{\theta_1}||||\phi_1(w+\bar{w}_\epsilon)||)\\ &= O(\exp(-CL)(\epsilon + \exp(-CL))) = O(\exp(-CL))\end{aligned}$$

and

$$f_0(u^L_{\theta_1}) = f_0(u_{\theta_1}) + O(||u^L_{\theta_1} - u_{\theta_1}||) = b + O(\exp(-CL)).$$

Hence

$$f_0(u^L_{\theta_1} + \phi_1(w+\bar{w}_\epsilon)) = b + O(\epsilon^2 + \exp(-CL)). \tag{3.21}$$

In addition

$$\begin{aligned}\epsilon(G(u^L_{\theta_1} + \phi_1(w+\bar{w}_\epsilon)) - G(u_{\theta_1})) &= \epsilon(G(u^L_{\theta_1}) - G(u_{\theta_1})) + \epsilon O(||\phi_1(w+\bar{w}_\epsilon)||)\\ &= \epsilon(O(||u_{\theta_1} - u^L_{\theta_1}||) + O(|w+\bar{w}_\epsilon|_\theta))\\ &= \epsilon O(\epsilon + \exp(-CL)) = O(\epsilon^2 + \exp(-C'L)).\end{aligned}$$

From (3.21) and the latter formula we get (3.20).

Combining (3.19) and (3.20) we derive

$$\beta(L,\epsilon,\theta_1,\dots,\theta_k) = f_\epsilon(S+(1-\phi_1)(w+\bar{w}_\epsilon)) - (k-1)b - \epsilon(G(u_{\theta_2}) + \ldots + G(u_{\theta_k})) + O(\exp(-CL)+\epsilon^2). \tag{3.22}$$

$(\theta_2,\dots,\theta_k)$ being fixed set $\gamma(\theta_1) = f_\epsilon(S+(1-\phi_1)(w+\bar{w}_\epsilon))$. In order to estimate $\gamma(\theta_1) - \gamma(\theta_1')$ we are going to compute $\frac{\partial\gamma}{\partial\theta_1}(\theta_1)$. Since $\frac{\partial}{\partial\theta_1}(\phi_1) = \frac{1}{2}\dot{\phi}_1$ we have:

$$\frac{\partial\gamma}{\partial\theta_1}(\theta_1) = (f_\epsilon'(S+(1-\phi_1)(w+\bar{w}_\epsilon)), -\frac{1}{2}\dot{\phi}_1(w+\bar{w}_\epsilon) + (1-\phi_1)\frac{\partial}{\partial\theta_1}(w+\bar{w}_\epsilon)).$$

Since supp $\dot{\phi}_1 \subset [-(\theta_1+\theta_2)/2-1, -(\theta_1+\theta_2)/2+1]$ we have that

$$|(f_\epsilon'(S+(1-\phi_1)(w+\bar{w}_\epsilon)), -\frac{1}{2}\dot{\phi}_1(w+\bar{w}_\epsilon))| \le C|f_\epsilon'(S+(1-\phi_1)(w+\bar{w}_\epsilon))|_\theta |\dot{\phi}_1(w+\bar{w}_\epsilon)|_\theta.$$

Therefore by lemmas 9 and 17

$$|(f'_\epsilon(S+(1-\phi_1)(w+\bar{w}_\epsilon)),-\frac{1}{2}\dot{\phi}_1(w+\bar{w}_\epsilon))| = O(\exp(-C(\theta_2-\theta_1))). \tag{3.23}$$

We can write:

$$(f'_\epsilon(S+(1-\phi_1)(w+\bar{w}_\epsilon)),(1-\phi_1)\frac{\partial}{\partial\theta_1}(w+\bar{w}_\epsilon)) = (f'_\epsilon(u^L_{\theta_1}+S+w+\bar{w}_\epsilon),(1-\phi_1)\frac{\partial}{\partial\theta_1}(w+\bar{w}_\epsilon))+s(\epsilon,\theta_1,\ldots,\theta_k)$$

where, by lemma 8-(ii),

$$|s(\epsilon,\theta_1,\ldots,\theta_k)| \le C\int_{\mathbb{R}}|\frac{d}{dt}(\phi_1(w+\bar{w}_\epsilon))||\frac{d}{dt}((1-\phi_1)\frac{\partial}{\partial\theta_1}(w+\bar{w}_\epsilon))|+|\phi_1(w+\bar{w}_\epsilon)||(1-\phi_1)\frac{\partial}{\partial\theta_1}(w+\bar{w}_\epsilon)|.$$

Since supp $\phi_1\cap$ supp $(1-\phi_1)\subset[-(\theta_1+\theta_2)/2-1,-(\theta_1+\theta_2)/2+1]$ we get by lemma 17

$$|s(\theta_1,\ldots,\theta_k,\epsilon)| = O(\exp(-C(\theta_2-\theta_1)))|\frac{\partial}{\partial\theta_1}(w+\bar{w}_\epsilon)|_\theta.$$

Hence by lemma 16

$$|s(\theta_1,\ldots,\theta_k,\epsilon)| = O(\exp(-C(\theta_2-\theta_1))). \tag{3.24}$$

Now we have $f'_\epsilon(u^L_{\theta_1}+S+w+\bar{w}_\epsilon)=\sum_{i=1}^k\tilde{\alpha}_{\epsilon,i}\dot{u}_{\theta_i}$. Using the properties of supp $\dot{u}^L_{\theta_i}$, supp ϕ_1, supp $(1-\phi_1)$ we derive

$$\begin{aligned}(f'_\epsilon(u_{\theta_1}+S+w+\bar{w}_\epsilon),(1-\phi_1)\frac{\partial}{\partial\theta_1}(w+\bar{w}_\epsilon)) &= \sum\nolimits_{i=1}^k\tilde{\alpha}_{\epsilon,i}(\dot{u}^L_{\theta_i},(1-\phi_1)\frac{\partial}{\partial\theta_1}(w+\bar{w}_\epsilon))\\ &= \sum\nolimits_{i=2}^k\tilde{\alpha}_{\epsilon,i}(\dot{u}^L_{\theta_i},\frac{\partial}{\partial\theta_1}(w+\bar{w}_\epsilon))\\ &= \sum\nolimits_{i=2}^k\tilde{\alpha}_{\epsilon,i}\frac{\partial}{\partial\theta_1}(\dot{u}^L_{\theta_i},w+\bar{w}_\epsilon)=0.\end{aligned}$$

Combining (3.23) and (3.24) we get:

$$\frac{\partial\gamma}{\partial\theta_1}(\theta_1)\le C\exp(-C'(\theta_2-\theta_1)).$$

Now let θ_1 and θ'_1 satisfy both $\theta_2-\theta_1>L$ and $\theta_2-\theta'_1>L$. We can assume that $\theta'_1<\theta_1$.

Then

$$\begin{aligned}|\gamma(\theta_1)-\gamma(\theta'_1)| &\le \int_{\theta'_1}^{\theta_1}|\frac{\partial\gamma(s)}{\partial\theta_1}|ds\\ &\le \int_{\theta'_1}^{\theta_1}C\exp(-C'(\theta_2-s))\,ds\\ &\le C''\exp(-C'(\theta_2-\theta_1)).\end{aligned}$$

Finally

$$|\gamma(\theta_1)-\gamma(\theta'_1)| = O(\exp(-CL)). \tag{3.25}$$

(3.22) and (3.25) imply the statement in lemma 18.□

We now show how to obtain k-bump solutions and then solutions with infinitely many bumps.

It is possible to prove that:

Theorem 3 *Let (V_1),(V_2),(W_1) and condition 1 hold. Then there exists a positive constant C_{17} such that: $\forall \omega > 0$ there exists $\epsilon_2 > 0$ such that $\forall \epsilon \in (-\epsilon_2, 0) \cup (0, \epsilon_2)$, $\forall k \in I\!N$, $\forall U_{i_1} = (c_{i_1}, d_{i_1}), \ldots, U_{i_k} = (c_{i_k}, d_{i_k})$ satisfying $\min_{l=1,\ldots,k-1}(c_{i_{l+1}} - d_{i_l}) > L_\epsilon := -C_{17}\ln|\epsilon|$ there are $(\theta_1, \ldots, \theta_k)$ with $\theta_l \in U_{i_l} = (c_{i_l}, d_{i_l})$ and a solution u_ϵ of (2.1) which satisfies:*

$$||u_\epsilon - \sum_{l=1}^{k} u_{\theta_l}^{L_\epsilon}||_{L^\infty(I\!R)} \leq \omega.$$

Proof: Choose $0 < \epsilon_2 < \epsilon_1$ such that for $|\epsilon| < \epsilon_2, \epsilon \neq 0$ and $L > L_\epsilon := -C_{17}\ln|\epsilon|$, where $C_{17} = 2/C_{16}$, one has in formula (3.18)

$$|O(\exp(-C_{16}L)) + \epsilon o_{L,\epsilon}(1)| < |\epsilon|\frac{\eta}{2}. \tag{3.26}$$

By lemma 13

$$\forall k,\ \forall(\theta_1, \ldots, \theta_k) \text{ such that } \min_i(\theta_{i+1} - \theta_i) > L \quad ||w(L, \theta_1, \ldots, \theta_k)||_\infty = O(\exp(-C_{10}L)) \tag{3.27}$$

and by lemma 14

$$||\bar{w}_\epsilon(L, \theta_1, \ldots, \theta_k)||_\infty \leq 2C_{12}|\epsilon|.$$

Hence for ϵ small enough and L large enough we have, for all $(\theta_1, \ldots, \theta_k)$ such that $\min_i(\theta_{i+1} - \theta_i) > L$,

$$||w(L, \theta_1, \ldots, \theta_k)||_\infty < \frac{\omega}{2} \quad \text{and} \quad ||\bar{w}_\epsilon(L, \theta_1, \ldots, \theta_k)||_\infty < \frac{\omega}{2}. \tag{3.28}$$

We may assume that we have chosen ϵ_2 small enough so that (3.28) is satisfied for all $|\epsilon| < \epsilon_2$ and $L > L_\epsilon$. Assume that $\min_l(c_{i_{l+1}} - d_{i_l}) > L_\epsilon$ for some $i_1 < \ldots < i_k$.

Define the function of k real variables $\bar{f}_\epsilon$ by $\bar{f}_\epsilon(\theta_1, ..., \theta_k) = f_\epsilon(u_{\theta_1}^{L_\epsilon} + \ldots + u_{\theta_k}^{L_\epsilon} + w(L_\epsilon, \theta_1, \ldots, \theta_k) + \bar{w}_\epsilon(L_\epsilon, \theta_1, \ldots, \theta_k))$. From now in this proof we assume, without loss of generality, that $\epsilon > 0$. $\bar{f}_\epsilon(\theta_1, ..., \theta_k)_{|\bar{U}_{i_1} \times \ldots \times \bar{U}_{i_k}}$ attains its minimum at some point $(\bar{\theta}_1, \ldots, \bar{\theta}_k) \in \bar{U}_{i_1} \times \ldots \times \bar{U}_{i_k}$. We claim that $(\bar{\theta}_1, \ldots, \bar{\theta}_k)$ is in $U_{i_1} \times \ldots \times U_{i_k}$. By lemma 15, it implies that $u_\epsilon = u_{\bar{\theta}_1}^{L_\epsilon} + \ldots + u_{\bar{\theta}_k}^{L_\epsilon} + w(L_\epsilon, \bar{\theta}_1, \ldots, \bar{\theta}_k) + \bar{w}_\epsilon(L_\epsilon, \bar{\theta}_1, \ldots, \bar{\theta}_k)$ is a solution of (2.1) which satisfies:

$$||u_\epsilon - \sum_{l=1}^{k} u_{\bar{\theta}_{i_l}}^{L_\epsilon}||_\infty \leq ||w||_\infty + ||\bar{w}_\epsilon||_\infty \leq \frac{\omega}{2} + \frac{\omega}{2} = \omega.$$

Let us prove that, for example, $\bar{\theta}_1 \neq d_{i_1}$. We argue by contradiction. If $\bar{\theta}_1 = d_{i_1}$, since $(d_{i_1}, \bar{\theta}_2, \ldots, \bar{\theta}_k)$ is a minimum of $\bar{f}_\epsilon$, we have:

$$\bar{f}_\epsilon(d_{i_1}, \bar{\theta}_2, \ldots, \bar{\theta}_k) - \bar{f}_\epsilon(a_{i_1}, \bar{\theta}_2, \ldots, \bar{\theta}_k) \leq 0. \tag{3.29}$$

On the other hand by formula (3.17) and from (3.26) we get:

$$\bar{f}_\epsilon(d_{i_1}, \bar{\theta}_2, \ldots, \bar{\theta}_k) - \bar{f}_\epsilon(a_{i_1}, \bar{\theta}_2, \ldots, \bar{\theta}_k) =$$

$$\epsilon(\Gamma(d_{i_1}) - \Gamma(a_{i_1})) + (\beta(d_{i_1}, \bar{\theta}_2, \ldots, \bar{\theta}_k) - \beta(a_{i_1}, \bar{\theta}_2, \ldots, \bar{\theta}_k)) \geq \epsilon\eta - \epsilon\frac{\eta}{2} = \epsilon\frac{1}{2}\eta > 0,$$

a contradiction with (3.29).

Similarly we can prove that $\bar{\theta}_1 \neq c_{i_1}$ and that for any l, $\bar{\theta}_l \notin \partial U_{i_l}$.

Therefore $\bar{f}_\epsilon$ has a minimum in $U_{i_1} \times \ldots \times U_{i_k}$ and the proof is complete.□

Remark 9 *By the exponential decay properties of u_0 the solution given by theorem 3 satisfies also the estimate :*

$$||u_\epsilon - \sum_{l=1}^{k} u_{\theta_{i_l}}||_{L^\infty(\mathbb{R})} \leq 2\omega,$$

provided ϵ is small enough.

Since L_ϵ does not depend on k it is possible to get from the above theorem, arguing as in [17], the existence of solutions with infinitely many bumps:

Theorem 4 *Let (V_1),(V_2),(W_1) and condition (1) hold. $\forall\omega > 0$, there is $\epsilon_2 > 0$ such that $\forall\epsilon \in (-\epsilon_2, 0)\cup(0, \epsilon_2)$, for any sequence of intervals $(U_{i_l} = (c_{i_l}, d_{i_l}))_{l\in J\subset\mathbb{Z}}$ satisfying $\inf_{l\in J}(c_{i_{l+1}} - d_{i_l}) > L_\epsilon = -C_{17}\ln|\epsilon|$, there are $(\theta_l)_{l\in J}$ with $\theta_l \in U_{i_l} = (c_{i_l}, d_{i_l})$ and a solution u_ϵ of (2.1) which satisfies:*

$$||u_\epsilon - \sum_{l\in J} u_{\theta_l}^{L_\epsilon}||_{L^\infty(\mathbb{R})} \leq \omega.$$

If J is infinite, such a solution u_ϵ has infinitely many bumps.

3.3 Solutions with bumps located near minima and maxima of Γ.

In this subsection we indicate a different condition on the Melnikov primitive which allows to find multibump homoclinic solutions of (2.1) with bumps located near maxima or minima of the Melnikov primitive. The details are omitted for the sake of brevity.

Assume that:

Condition 2 *There are $\eta > 0$ and a sequence $(U_n = (c_n, d_n))_{n\in\mathbb{Z}}$ of bounded open intervals of $\mathbb{R}$ which satisfy:*

(i) *For any n, either "$\Gamma'(c_n) > \eta$ and $\Gamma'(d_n) < -\eta$" or "$\Gamma'(c_n) < -\eta$ and $\Gamma'(d_n) > \eta$";*

(ii) *$c_n \to +\infty$ as $n \to +\infty$ and $d_n \to -\infty$ as $n \to -\infty$.*

Theorem 5 *Let (V_1),(V_2),(W_1) and condition 2 hold. Assume that D^2V is locally Lipschitz continuous. Then the statements of theorems 3 and 4 hold.*

We will not give the proof of theorem 5. We just specify that it is enough to prove that $\forall|\epsilon| < \epsilon_1$, $\forall k$, $\forall(\theta_1, \ldots, \theta_k) \in \mathbb{R}^k$ with $\min_i(\theta_i - \theta_{i-1}) > L$ one has:

$$\left|\frac{\partial\beta}{\partial\theta_i}(L, \epsilon, \theta_1, \ldots, \theta_k)\right| = O(\exp(-CL)) + \epsilon o_{\epsilon,L}(1), \qquad i = 1, \ldots, k,$$

where β is defined by formula (3.17). This can be done thanks to the estimates given in lemmas 9, 10, 13, 14 and 16. Then a simple degree argument yields theorem 5.

As a consequence of theorem 5 it is possible to prove the existence of solutions with infinitely many bumps, these bumps being located near maxima and minima of the Melnikov primitive.

3.4 Non-degeneracy of the Melnikov primitive and a uniqueness result

In this section we show that if the Melnikov primitive Γ possesses non-degenerate critical points then a uniqueness result can be proved.

To make a precise statement we assume the following condition:

Condition 3 *There exist $\eta, \nu > 0$ and a sequence $(a_n)_{n\in\mathbb{Z}}$ of critical points of the Melnikov primitive Γ with $a_n \to +\infty$ as $n \to +\infty$ and $a_n \to -\infty$ as $n \to -\infty$ such that for any n one has:*

$$\min_{\theta\in[a_n-\nu,a_n+\nu]}|\Gamma''(\theta)| \geq \eta.$$

Remark 10 *Note that condition 3 implies condition 1 (if $\Gamma'' \geq \eta$ on $[a_n-\nu, a_n+\nu]$) as well as condition 2, with $(c_n, d_n) = (a_n - \nu, a_n + \nu)$.*

Remark 11 *If Γ is periodic condition 3 is satisfied whenever Γ possesses at least one non-degenerate critical point $\bar{a}$: take $a_n = \bar{a} + Tn$, where T is the period.*

The main result of this subsection is the following theorem:

Theorem 6 *Let (V_1),(V_2),(W_1) and condition 3 hold. Moreover assume that D^2V is locally Lipschitz continuous. Then there exist $\omega_3 > 0, \epsilon_3 > 0$ and a positive constant C_{18} such that: for all $\epsilon \in (-\epsilon_3, 0) \cup (0, \epsilon_3)$, for all $k \in \mathbb{N}$, for all (possibly infinite) sequence $(i_q)_{q\in J\subset\mathbb{Z}}$ such that $a_{i_{q+1}} - a_{i_q} > L'_\epsilon + 2\nu := -C_{18}\ln|\epsilon| + 2\nu$ there is a unique solution u_ϵ of (2.1) which satisfies:*

$$||u_\epsilon - \sum_{q\in J} u^{L'_\epsilon}_{a_{i_q}}||_{L^\infty(\mathbb{R})} \leq \omega_3.$$

To prove theorem 6 we need the following lemmas:

Lemma 19 *Assume that D^2V is locally Lipschitz continuous. Then there is a positive constant C_{19} such that for all $L > L_2$ and $|\epsilon| \leq \epsilon_1$, for all k and for all $(\theta_1, ..., \theta_k) \in \mathbb{R}^k$ with $\min_i(\theta_{i+1}-\theta_i) > L$*

$$|\frac{\partial}{\partial\theta_i}(f'_\epsilon(u^L_{\theta_1} + \ldots + u^L_{\theta_k} + w + \bar{w}_\epsilon), \dot{u}^L_{\theta_i}) - \epsilon\Gamma''(\theta_i)| = o_{L,\epsilon}(1)(\epsilon + \exp(-C_{19}L))$$

for any $i = 1, \ldots, k$.

Proof: We have:

$$\begin{aligned} \frac{\partial}{\partial\theta_i}(f'_\epsilon(u^L_{\theta_1} + \ldots + u^L_{\theta_k} + w + \bar{w}_\epsilon), \dot{u}^L_{\theta_i}) &= \\ (f''_\epsilon(u^L_{\theta_1} + \ldots + u^L_{\theta_k} + w + \bar{w}_\epsilon)\dot{u}^L_{\theta_i}, \dot{u}^L_{\theta_i} + \frac{\partial}{\partial\theta_i}(w + \bar{w}_\epsilon)) &+ \\ (f'_\epsilon(u^L_{\theta_1} + \ldots + u^L_{\theta_k} + w + \bar{w}_\epsilon), \ddot{u}^L_{\theta_i}) &= \\ (f''_\epsilon(u^L_{\theta_i} + w + \bar{w}_\epsilon)\dot{u}^L_{\theta_i}, \dot{u}^L_{\theta_i} + \frac{\partial}{\partial\theta_i}(w + \bar{w}_\epsilon)) + (f'_\epsilon(u^L_{\theta_i} + w + \bar{w}_\epsilon), \ddot{u}^L_{\theta_i}) &= U_1 + U_2 \end{aligned}$$

where:

$$U_1 = (f''_0(u^L_{\theta_i} + w + \bar{w}_\epsilon)\dot{u}^L_{\theta_i}, \dot{u}^L_{\theta_i} + \frac{\partial}{\partial\theta_i}(w + \bar{w}_\epsilon)) + (f'_0(u^L_{\theta_i} + w + \bar{w}_\epsilon), \ddot{u}^L_{\theta_i})$$

and

$$U_2 = \epsilon(G''(u^L_{\theta_i} + w + \bar{w}_\epsilon)\dot{u}^L_{\theta_i}, \dot{u}^L_{\theta_i} + \frac{\partial}{\partial\theta_i}(w + \bar{w}_\epsilon)) + \epsilon(G'(u^L_{\theta_i} + w + \bar{w}_\epsilon), \ddot{u}^L_{\theta_i}).$$

We now estimate U_2. Lemma 9-(iv) implies that:

$$|(G''(u^L_{\theta_i} + w + \bar{w}_\epsilon)\dot{u}^L_{\theta_i}, \frac{\partial}{\partial\theta_i}(w + \bar{w}_\epsilon))| \leq C|\frac{\partial}{\partial\theta_i}(w + \bar{w}_\epsilon)|_\theta.$$

Hence by lemma 16:

$$U_2 = \epsilon(G''(u^L_{\theta_i} + w + \bar{w}_\epsilon)\dot{u}^L_{\theta_i}, \dot{u}^L_{\theta_i}) + \epsilon(G'(u^L_{\theta_i} + w + \bar{w}_\epsilon), \ddot{u}^L_{\theta_i}) + \epsilon o_{\epsilon,L}(1).$$

Now we have:

$$(G''(u^L_{\theta_i} + w + \bar{w}_\epsilon)\dot{u}^L_{\theta_i}, \dot{u}^L_{\theta_i}) = \int_{\mathbb{R}} -D^2W(t, u^L_{\theta_i} + w + \bar{w}_\epsilon)\dot{u}^L_{\theta_i}\dot{u}^L_{\theta_i}.$$

By lemmas 10-(i), 13, 14

$$||u^L_{\theta_i} + w + \bar{w}_\epsilon - u_{\theta_i}||_\infty = o_{L,\epsilon}(1).$$

We easily see by (W_1) that this implies

$$(G''(u^L_{\theta_i} + w + \bar{w}_\epsilon)\dot{u}^L_{\theta_i}, \dot{u}^L_{\theta_i}) = (G''(u_{\theta_i})\dot{u}^L_{\theta_i}, \dot{u}^L_{\theta_i}) + o_{L,\epsilon}(1).$$

In the same way

$$(G'(u^L_{\theta_i} + w + \bar{w}_\epsilon), \ddot{u}^L_{\theta_i}) = (G'(u_{\theta_i}), \ddot{u}^L_{\theta_i}) + o_{L,\epsilon}(1).$$

Finally since: $||\dot{u}^L_{\theta_i} - \dot{u}_{\theta_i}||_{L^2} = ||\dot{u}^L_0 - \dot{u}_0||_{L^2} \to 0$ and $||\ddot{u}^L_{\theta_i} - \ddot{u}_{\theta_i}||_{L^2} = ||\ddot{u}^L_0 - \ddot{u}_0||_{L^2} \to 0$ as $L \to +\infty$ we have, by lemma 9-(ii)-(iv):

$$U_2 = \epsilon(G''(u_{\theta_i})\dot{u}_{\theta_i}, \dot{u}_{\theta_i}) + \epsilon(G'(u_{\theta_i}), \ddot{u}_{\theta_i}) + \epsilon o_{L,\epsilon}(1) = \epsilon\Gamma''(\theta_i) + \epsilon o_{L,\epsilon}(1). \tag{3.30}$$

In order to estimate U_1 we use the invariance of f_0 under the action of $\mathbb{R}$. In fact this property implies that:

$$\forall v_1, v_2 \in E,\ \forall \theta \in \mathbb{R} \qquad (f_0'(\theta * v_1), \theta * v_2) = (f_0'(v_1), v_2).$$

Deriving with respect to θ we get:

$$({f_0}''(v_1)\dot{v}_1, v_2) + (f_0'(v_1), \dot{v}_2) = 0. \tag{3.31}$$

Applying (3.31) to $v_1 = u^L_{\theta_i} + w + \bar{w}_\epsilon$ and $v_2 = \dot{u}^L_{\theta_i}$ we obtain:

$$\begin{aligned} U_1 &= (f_0''(u^L_{\theta_i} + w + \bar{w}_\epsilon)\dot{u}^L_{\theta_i}, \dot{u}^L_{\theta_i} + \frac{\partial}{\partial\theta_i}(w + \bar{w}_\epsilon)) \\ &- (f_0''(u^L_{\theta_i} + w + \bar{w}_\epsilon)\dot{u}^L_{\theta_i}, \dot{u}^L_{\theta_i} + \dot{w} + \dot{\bar{w}}_\epsilon) \\ &= (f_0''(u^L_{\theta_i} + w + \bar{w}_\epsilon)\dot{u}^L_{\theta_i}, \frac{\partial}{\partial\theta_i}(w + \bar{w}_\epsilon) - (\dot{w} + \dot{\bar{w}}_\epsilon)). \end{aligned}$$

Now by lemma 10-(iii) and since supp $\dot{u}^L_{\theta_i} \subset I_i = [-(\theta_i + \theta_{i+1})/2, -(\theta_i + \theta_{i-1})/2]$, we can write for any $v \in E$

$$|(f_0''(u^L_{\theta_i} + w + \bar{w}_\epsilon)\dot{u}^L_{\theta_i}, v)| = O(|w + \bar{w}_\epsilon|_\theta + \exp(-C_8 L))|v|_\theta.$$

Hence by lemmas 13, 14

$$|U_1| = O(\epsilon + \exp(-CL))(|\frac{\partial}{\partial\theta_i}(w + \bar{w}_\epsilon)|_\theta + |\dot{w} + \dot{\bar{w}}_\epsilon|_\theta). \tag{3.32}$$

By lemma 16 we have that:

$$|\frac{\partial}{\partial\theta_i}(w + \bar{w}_\epsilon)|_\theta = O(\epsilon + \exp(-C_{14} L)). \tag{3.33}$$

Now we have to estimate $|\dot{w} + \dot{\bar{w}}_\epsilon|_\theta = |\dot{\tilde{w}}_\epsilon|_\theta$.
Since we already have an estimate of $|w + \bar{w}_\epsilon|_\theta$ we have only to prove that

$$\sup_i \left(\int_{I_i} \ddot{\tilde{w}}_\epsilon(L,\theta)^2 \right) = o_L(1) + O(\epsilon^2).$$

We have by lemma 14

$$f'_\epsilon(\sum_{i=1}^k u^L_{\theta_i} + \tilde{w}_\epsilon(L,\theta)) = \sum_{i=1}^k \tilde{\alpha}_i(L,\epsilon,\theta)\dot{u}^L_{\theta_i}$$

with $|\tilde{\alpha}_i(L,\epsilon,\theta)| = O(\epsilon + \exp(-C_{13}L))$.
Hence on I_i

$$-\ddot{u}^L_{\theta_i} + u^L_{\theta_i} - \ddot{\tilde{w}}_\epsilon + \tilde{w}_\epsilon - \nabla V(u^L_{\theta_i} + \tilde{w}_\epsilon) - \epsilon \nabla W(t, u^L_{\theta_i} + \tilde{w}_\epsilon) = \tilde{\alpha}_{\epsilon,i}(- \overset{...}{u}{}^L_{\theta_i} + \dot{u}^L_{\theta_i}).$$

Consequently on I_i

$$|\ddot{\tilde{w}}_\epsilon| \le |\tilde{w}_\epsilon| + |-\ddot{u}^L_{\theta_i} + u^L_{\theta_i} - \nabla V(u^L_{\theta_i})| + |\nabla V(u^L_{\theta_i}) - \nabla V(u^L_{\theta_i} + \tilde{w}_\epsilon)| + |\tilde{\alpha}_{\epsilon,i}||- \overset{...}{u}{}^L_{\theta_i} + \dot{u}^L_{\theta_i}| + \epsilon|\nabla W(t, u^L_{\theta_i} + \tilde{w}_\epsilon)|.$$

By (V_1) and (W_1) $|\nabla V(u^L_{\theta_i}) - \nabla V(u^L_{\theta_i} + \tilde{w}_\epsilon)| \le C|\tilde{w}_\epsilon|$ and $|\nabla W(t, u^L_{\theta_i} + \tilde{w}_\epsilon)| \le C(|u^L_{\theta_i}| + |\tilde{w}_\epsilon|)$.
Moreover

$$|-\ddot{u}^L_{\theta_i} + u^L_{\theta_i} - \nabla V(u^L_{\theta_i})| = |-(\ddot{u}^L_{\theta_i} - \ddot{u}_{\theta_i}) + (u^L_{\theta_i} - u_{\theta_i}) - (\nabla V(u^L_{\theta_i}) - \nabla V(u_{\theta_i}))| \le |\ddot{u}^L_{\theta_i} - \ddot{u}_{\theta_i}| + C'|u^L_{\theta_i} - u_{\theta_i}|.$$

Hence on I_i

$$|\ddot{\tilde{w}}_\epsilon| \le C''(|\tilde{w}_\epsilon| + \epsilon|u^L_{\theta_i}| + |\ddot{u}^L_{\theta_i} - \ddot{u}_{\theta_i}| + |u^L_{\theta_i} - u_{\theta_i}|) + |\tilde{\alpha}_{\epsilon,i}|(|\overset{...}{u}{}^L_{\theta_i}| + |\dot{u}^L_{\theta_i}|).$$

Now lemmas 14 (estimates on $|\tilde{w}_\epsilon|_\theta$ and on $|\tilde{\alpha}_{\epsilon,i}|$) and 9 (estimates on $||\dot{u}^L_{\theta_i} - \dot{u}_{\theta_i}||$ and on $||u^L_{\theta_i} - u_{\theta_i}||$) can be used to get

$$|\ddot{\tilde{w}}_\epsilon|_\theta = O(\epsilon + \exp(-CL)). \tag{3.34}$$

(3.34) combined with (3.30), (3.32) and (3.33) yields the desired result.

Lemma 20 *Under the same hypotheses as in lemma 19 there is a positive constant C_{20} such that for all $l \in \{1,\ldots,k\}$, for all $\lambda_1,\ldots,\lambda_k \in I\!R^k$*

$$\left| \sum_{j=1}^k \lambda_j \frac{\partial}{\partial \theta_j}(f'_\epsilon(u^L_{\theta_1} + \ldots + u^L_{\theta_k} + \tilde{w}_\epsilon), \dot{u}^L_{\theta_l}) - \lambda_l \epsilon \Gamma''(\theta_l) \right| = \sup_j |\lambda_j|(\epsilon o_{L,\epsilon}(1) + O(\exp(-C_{20}L))).$$

Proof:
By lemma 19 it is enough to prove that:

$$|U| = \sup_j |\lambda_j|(\epsilon o_{L,\epsilon}(1) + O(\exp(-C_{20}L))$$

where

$$U = \sum_{j\ne l} \lambda_j \frac{\partial}{\partial \theta_j}(f'_\epsilon(u^L_{\theta_1} + \ldots + u^L_{\theta_k} + \tilde{w}_\epsilon), \dot{u}^L_{\theta_l}) = \sum_{j\ne l} \lambda_j (f''_\epsilon(\sum_{i=1}^k u^L_{\theta_i} + \tilde{w}_\epsilon)(\dot{u}^L_{\theta_j} + \frac{\partial \tilde{w}_\epsilon}{\partial \theta_j}), \dot{u}^L_{\theta_l}).$$

Now since supp $\dot{u}^L_{\theta_l} \cap$ supp $\dot{u}^L_{\theta_j} = \emptyset$ for $j \neq l$ we have

$$U = (f''_\epsilon(u^L_{\theta_l} + \tilde{w}_\epsilon)\dot{u}^L_{\theta_l}, \sum_{j\neq l} \lambda_j \frac{\partial \tilde{w}_\epsilon}{\partial \theta_j}).$$

Let $\psi \in E$ satisfy $\psi = 0$ outside I_l, $\psi = 1$ on $(-\theta_l - L/4, -\theta_l + L/4)$, ψ is continuous and linear on each component of $I_l \backslash (-\theta_l - L/4, -\theta_l + L/4)$. Then, since supp $u^L_{\theta_l} \subset (-\theta_l - L/4, -\theta_l + L/4)$,

$$\begin{aligned} |(f''_\epsilon(u^L_{\theta_l} + \tilde{w}_\epsilon)\dot{u}^L_{\theta_l}, \sum_{j\neq l} \lambda_j \frac{\partial \tilde{w}_\epsilon}{\partial \theta_j})| &= |(f''_\epsilon(u^L_{\theta_l} + \tilde{w}_\epsilon)\dot{u}^L_{\theta_l}, \psi(\sum_{j\neq l} \lambda_j \frac{\partial \tilde{w}_\epsilon}{\partial \theta_j}))| \\ &\leq C|f''_\epsilon(u^L_{\theta_l} + \tilde{w}_\epsilon)\dot{u}^L_{\theta_l}|_{W^{1,2}(I_l)}|\sum_{j\neq l} \lambda_j \frac{\partial \tilde{w}_\epsilon}{\partial \theta_j}|_{W^{1,2}(I_l)} \\ &\leq C|f''_\epsilon(u^L_{\theta_l} + \tilde{w}_\epsilon)\dot{u}^L_{\theta_l}|_\theta|\sum_{j\neq l} \lambda_j \frac{\partial \tilde{w}_\epsilon}{\partial \theta_j}|_\theta. \end{aligned}$$

Now by lemma 16

$$\left|\sum_{j\neq l} \lambda_j \frac{\partial \tilde{w}_\epsilon}{\partial \theta_j}\right|_\theta = \max_j |\lambda_j| O(\epsilon + \exp(-C_{14}L)).$$

Moreover by lemmas 10-(iii) and 9-(iv)

$$|f''_\epsilon(u^L_{\theta_l} + \tilde{w}_\epsilon)\dot{u}^L_{\theta_l}|_\theta \leq |f''_0(u^L_{\theta_l} + \tilde{w}_\epsilon)\dot{u}^L_{\theta_l}|_\theta + |\epsilon||G''(u^L_{\theta_l} + \tilde{w}_\epsilon)\dot{u}^L_{\theta_l}|_\theta = O(\epsilon + \exp(-C_8 L)).$$

So we get the desired estimate $\Box$.

Proof of theorem 6:

We shall prove theorem 6 in the case where $J = \mathbb{Z}$ (so the sequence i_l is infinite). The existence of the solution u_ϵ is a consequence of theorems 4 or 5: ω_3 being fixed, choose $\bar{\nu} < \nu$ small enough so that for all $L > 8$, $\forall k$, for all sequences $(\theta_{i_q})_{q\in\mathbb{Z}}$ with $|\theta_{i_q} - a_{i_q}| < \bar{\nu}$ and $\theta_{i_{q+1}} - \theta_{i_q} > L$:

$$||\sum_{q\in\mathbb{Z}} (u^L_{\theta_{i_q}} - u^L_{a_{i_q}})||_\infty \leq \frac{\omega_3}{2} \tag{3.35}$$

and apply theorem 5 with $U_i = (a_i - \bar{\nu}, a_i + \bar{\nu})$. We get that for $\epsilon \neq 0$ small enough there exists $L_\epsilon = -C\ln|\epsilon|$ such that if $\min_q(a_{i_{q+1}} - a_{i_q}) > L_\epsilon$ then there exists a solution of (2.1) u_ϵ with:

$$||u_\epsilon - \sum_{q\in\mathbb{Z}} u^{L_\epsilon}_{\theta_{i_q}}||_\infty \leq \frac{\omega_3}{2} \tag{3.36}$$

(3.35) and (3.36) imply that:

$$||u_\epsilon - \sum_{q\in\mathbb{Z}} u^{L_\epsilon}_{a_{i_q}}||_\infty \leq \omega_3.$$

In order to prove the uniqueness take ϵ_3 small enough so that for $|\epsilon| < \epsilon_3, \epsilon \neq 0$ and $L > L'_\epsilon := (-2/C_{20}) \ln|\epsilon|$ we have in lemma 20

$$|\epsilon o_{L,\epsilon}(1) + O(\exp(-C_{20}L))| < |\epsilon|\eta/3. \tag{3.37}$$

An easy consequence of lemma 11 is that there is $\delta_3 \in (0, \delta_1)$ such that for all $L > 8$ for all sequence $(i_q)_{q\in\mathbb{Z}}$ such that $a_{i_{q+1}} - a_{i_q} > L + 2\nu$, if $|v - \sum_{q\in\mathbb{Z}} u^L_{a_{i_q}}|_a < \delta_3$ then there are $(\theta_{i_q})_{q\in\mathbb{Z}}$ and $w \in W^{1,2}_{loc}$ which satisfy:

$$v = \sum_{q\in\mathbb{Z}} u^L_{\theta_{i_q}} + w \ , \quad (w, \dot{u}^L_{\theta_{i_q}}) = 0, \tag{3.38}$$

$$\theta_{i_q} \in (a_{i_q} - \nu, a_{i_q} + \nu)\ , \quad |w|_\theta < \delta_1. \tag{3.39}$$

Here $|\cdot|_a$ and $|\cdot|_\theta$ denote the norms associated respectively with (a_{i_q}) and θ_{i_q} by remark 8.

Now, by standard regularity properties for the solutions of (2.1), there is $\omega_3 > 0$ such that if v is a solution of (2.1) then

$$||v - \sum_{q\in\mathbb{Z}} u^L_{a_{i_q}}||_\infty < \omega_3 \Longrightarrow |v - \sum_{q\in\mathbb{Z}} u^L_{a_{i_q}}|_a < \delta_3$$

provided L is large enough and ϵ is small enough.

Arguing by contradiction we assume that for some $\epsilon \in (-\epsilon_3, \epsilon_3)$ there exist two different solutions of (2.1), u_ϵ and u'_ϵ which are at a distance smaller than ω_3 from $\sum_{q\in\mathbb{Z}} u^{L'_\epsilon}_{a_{i_q}}$ with $a_{i_{q+1}} - a_{i_q} > L'_\epsilon + 2\nu$. Fix $L = L'_\epsilon$. Then by the above arguments we can write (provided ϵ_3 has been chosen small enough)

$$u_\epsilon = \sum_{q\in\mathbb{Z}} u^L_{\theta_{i_q}} + w_\epsilon, \quad u'_\epsilon = \sum_{q\in\mathbb{Z}} u^L_{\theta'_{i_q}} + w'_\epsilon,$$

with $|\theta_{i_q} - a_{i_q}| < \nu$, $|\theta'_{i_q} - a_{i_q}| < \nu$, $(w_\epsilon, \dot u^L_{\theta_{i_q}}) = 0$, $(w'_\epsilon, \dot u^L_{\theta'_{i_q}}) = 0$. By remark 8, if $\theta_{i_q} = \theta'_{i_q}$ for all q then $w_\epsilon = w'_\epsilon$. Hence there is l such that $\theta_{i_l} \neq \theta'_{i_l}$. We can assume that

$$|\theta_{i_l} - \theta'_{i_l}| \geq \sup_q |\theta_{i_q} - \theta'_{i_q}|/2.$$

Set $w_k = \tilde w_\epsilon(L, \theta_{i_{-k}}, \ldots, \theta_{i_k})$ and $w'_k = \tilde w_\epsilon(L, \theta_{i_{-k}}, \ldots, \theta_{i_k})$. We have

$$f'_\epsilon(\sum_{q=-k}^{k} u^L_{\theta_{i_q}} + w_k) = \sum_{q=-k}^{k} \tilde\alpha_{\epsilon, i_q} \dot u^L_{\theta_{i_q}}$$

and it is easy to see that this equation implies, by lemma 14, that the sequence w_k is precompact in $W^{1,2}_{loc}$. Now, if a subsequence of w_k converges to $w \in W^{1,2}_{loc}$ then we must have

$$f'_\epsilon(\sum_{q\in\mathbb{Z}} u^L_{\theta_{i_q}} + w) = \sum_{q\in Z} \alpha_{i_q} \dot u^L_{\theta_{i_q}} \quad , \quad |w|_\theta < \delta_1.$$

for some (α_{i_q}), which implies by remark 8 that $w = w_\epsilon$. Hence w_k converges to w_ϵ in $W^{1,2}_{loc}$. Similarly w'_k converges to w'_ϵ. Therefore

$$\sum_{q=-k}^{k} u^L_{\theta_{i_q}} + w_k \to u_\epsilon \quad \text{and} \quad \sum_{q=-k}^{k} u^L_{\theta'_{i_q}} + w'_k \to u'_\epsilon \quad \text{in } W^{1,2}_{loc}$$

By remark 8 it implies that

$$(f'_\epsilon(\sum_{q=-k}^{k} u^L_{\theta_{i_q}} + w_k), \dot u^L_{\theta_{i_l}}) \to 0 \text{ and } (f'_\epsilon(\sum_{q=-k}^{k} u^L_{\theta'_{i_q}} + w'_k), \dot u^L_{\theta'_{i_l}}) \to 0$$

as $k \to \infty$. Set $\lambda_{i_q} = \theta'_{i_q} - \theta_{i_q}$. Let g_k be the function defined on $[0,1]$ by

$$g(s) = (f'_\epsilon(\sum_{q=-k}^{k} u^L_{\theta_{i_q} + s\lambda_{i_q}} + \tilde w_\epsilon(L, \theta_{i_{-k}} + s\lambda_{i_{-k}}, \ldots, \theta_{i_k} + s\lambda_{i_k})), \dot u^L_{\theta_{i_l} + s\lambda_{i_l}}).$$

We have $\lim_{k\to\infty} g_k(0) = \lim_{k\to\infty} g_k(1) = 0$. Hence there exist points $s_k \in [0,1]$ such that $g'_k(s_k) \to 0$. This implies

$$\sum\nolimits_{q=-k}^{k} \lambda_{i_q} \frac{\partial}{\partial \theta_{i_q}} (f'_\epsilon(\sum_{q=-k}^{k} u^L_{\theta_{i_q}+s_k\lambda_{i_q}} + \tilde{w}_\epsilon), \dot{u}^L_{\theta_{i_l}+s_k\lambda_{i_l}}) \to 0. \tag{3.40}$$

Hence by lemma 20 we deduce that

$$|\lambda_{i_l}\epsilon\Gamma''(\theta_{i_l}+s_k\lambda_{i_l})| \le |2\lambda_{i_l}|(\epsilon o_{L,\epsilon}(1) + O(\exp(-C_{20}L)) + o_k(1)$$

with $\lim_{k\to\infty} o_k(1) = 0$. By Condition 3 and (3.37) we derive that:

$$|\epsilon|\eta < |\epsilon\Gamma''(\theta_{i_j}+s_k\lambda_{i_j})| \le 2|\epsilon o_{L,\epsilon}(1) + O(\exp(-CL))| \le |\epsilon|\frac{2\eta}{3}.$$

This contradiction concludes the proof of theorem 6. □

4 Bernoulli shift

In this section we describe some consequences of the previous results when the perturbation W is T-periodic in time.

First of all let us recall some well-known results. Consider a diffeomorphism $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ with a hyperbolic fixed point p. To make a precise description of the dynamics of Φ in presence of a transverse homoclinic point $r \neq p$ we recall the definition of an abstract Bernoulli shift structure.

Let us consider the space $\Sigma = \{0,1\}^{\mathbb{Z}}$. Σ is endowed with the standard metric:

$$d(s,\bar{s}) = \sum\nolimits_{n=-\infty}^{n=+\infty} \frac{|s_n - \bar{s}_n|}{2^n}.$$

With such a metric Σ is compact, totally disconnected and perfect, *i.e* Σ is a Cantor set.

On Σ acts the continuous shift-map σ defined by $(\sigma(s))_i = s_{i+1}$. The shift-map σ is the prototype of chaotic map. Indeed σ has a countable infinity of periodic orbits of arbitrarily high periods, an uncountable infinity of non-periodic orbits and a dense orbit; σ exhibits sensitive dependence on initial conditions.

The Smale-Birkhoff theorem states that if $r \neq p$ is a point of transverse intersection between the stable and the unstable manifold of p then there are $l \in \mathbb{N}$ and and a homeomorphism $\tau : \{0,1\}^{\mathbb{Z}} \to I \subset \mathbb{R}^d$, where $I = \tau(\Sigma)$ is an invariant Cantor set for Φ^l, such that $\Phi^l \circ \tau = \tau \circ \sigma$.

In particular the Smale-Birkhoff theorem implies that the map Φ exhibits sensitive dependence on initial conditions. In fact such a theorem implies a stronger property, namely that the topological entropy h_{top} of Φ is positive. Let us recall that h_{top} is defined by the following expression:

$$h_{top} = \sup_{R>0}\lim_{e\to 0}(\limsup_{n\to\infty} \frac{\ln s(n,e,R)}{n})$$

where:

$$s(n,e,R) = \max\{\mathrm{Card}(E) : E \subset B(0,R) \mid \forall x \neq y \in E \ \max_{0\le k\le n} |\Phi^k(x) - \Phi^k(y)| \ge e\}.$$

h_{top} is a measure of the asymptotic distortion of the iterates of Φ along the orbits. For example the topological entropy of an isometry Φ is 0.

In small perturbations of autonomous Hamiltonian systems the transversality condition required by the Smale-Birkhoff theorem can be checked using the Melnikov function. Indeed the Melnikov function Γ' measures perturbatively the distance between W^u and W^s; hence non-degenerate critical points of the Melnikov primitive correspond to transverse intersections between W^s and W^u.

Now we are able to describe some consequences of the results of the previous sections when the perturbation $W(\cdot, u)$ is T-periodic. In this case one can define the Poincaré map $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by $\Phi(x_0, \dot{x}_0) = (x(T, x_0, \dot{x}_0), \dot{x}(T, x_0, \dot{x}_0))$ where $x(t, x_0, \dot{x}_0)$ is the solution of (2.1) which satisfies the initial conditions $x(0, x_0, \dot{x}_0) = x_0$, $\dot{x}(0, x_0, \dot{x}_0) = \dot{x}_0$. From (V_1) the point $0 \in \mathbb{R}^{2n}$ is a hyperbolic fixed point for the map Φ.

4.1 Approximate Bernoulli shift

A variational approach to the study of chaotic behaviours in Hamiltonian systems through variational methods started with the work of E.Séré [17]. He proved that the existence of solutions with infinitely many bumps implies that it is possible to embed in the dynamics of the system an approximate (discontinuous) Bernoulli shift structure and that the topological entropy of the system is positive. (See [17] for a precise definition of an approximate Bernoulli shift). In particular in [17] the estimate on the topological entropy $h_{top} \geq C/L$ is given, where C is a positive constant and L is the sufficient distance between two adjacent bumps to glue up the bumps.

The same arguments of [17] show that theorem 4 or 5 implies the existence in system (2.1) of an approximate Bernoulli shift structure and that the topological entropy h_{top} is positive for $\epsilon \neq 0$ small enough. In particular we give the following estimate from below for h_{top}:

$$h_{top} \geq \frac{C}{L_\epsilon} = -\frac{C}{C_{17} \ln |\epsilon|}.$$

However, if we do not have any uniqueness property in theorems 4 or 5, it does not seem possible to get a complete (continuous) Bernoulli shift in general.

4.2 Complete Bernoulli shift

According to section 3.3, uniqueness can be obtained by the non-degeneracy of the critical points of the Melnikov primitive. This implies that it is possible to embed a continuous Bernoulli shift in the dynamics of the system. Thus we can prove by our method the following classical result:

Theorem 7 *Let condition (V_1),(V_2) and (W_1) hold. Moreover suppose that W is 1-periodic in time and that D^2V is locally Lipschitz. If the Melnikov primitive Γ possesses at least one non-degenerate critical point $\bar{a}$ then for $\epsilon \neq 0$ small enough there exist $l \in \mathbb{N}$ and a homeomorphism $\tau : \Sigma = \{0,1\}^{\mathbb{Z}} \to I \subset \mathbb{R}^{2n}$ onto its image such that the following diagram:*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\tau} & \mathbb{R}^{2n} \\ \sigma\big\downarrow & & \big\downarrow \Phi^l \\ \Sigma & \xrightarrow{\tau} & \mathbb{R}^{2n} \end{array}$$

commutes.

Proof:

Fix a positive constant $r > 2||u_0||_\infty$. Consider the set:

$$\Lambda_r = \{u \in L^\infty(\mathbb{R}) \mid u \text{ solves (2.1) and } ||u||_\infty \le r\}.$$

Λ_r equipped with the topology of the uniform convergence on compact subsets of $\mathbb{R}$ is a compact, metrizable topological space.

Let $\omega_4 = \min(\omega_3, ||u_0||_\infty/6)$, where ω_3 is given by theorem 6.

By theorem 6 there exists $\epsilon_3 > 0$ such that for any $|\epsilon| < \epsilon_3$, there is L'_ϵ such that for any infinite sequence $s = \{\dots, s_{-n}, \dots, s_{-1}, s_0, s_1, \dots, s_n, \dots\} \in \{0,1\}^{\mathbb{Z}}$ there is a unique solution u_ϵ of (2.1) such that:

$$||u_\epsilon - \sum_{i=-\infty}^{i=+\infty} s_i u_{a_i}^{L'_\epsilon}||_\infty < \omega_4$$

where $a_i = \bar{a} + i([L'_\epsilon] + 1)$.

Consider the map:

$$J : \{0,1\}^{\mathbb{Z}} \to \Lambda_r$$

which assigns to any $s = \{\dots, s_{-n}, \dots, s_{-1}, s_0, s_1, \dots, s_n, \dots\} \in \{0,1\}^{\mathbb{Z}}$ the unique solution $J(s) = u_\epsilon$.

Note also that if $s = \{\dots, s_{-n}, \dots, s_{-1}, s_0, s_1, \dots, s_n, \dots\} \ne \bar{s} = \{\dots, \bar{s}_{-n}, \dots, \bar{s}_{-1}, \bar{s}_0, \bar{s}_1, \dots, \bar{s}_n, \dots\}$ then:

$$||\sum_i s_i u_{a_i}^L - \sum_i \bar{s}_i u_{a_i}^L||_\infty \ge ||u_0^L||_\infty, \tag{4.1}$$

hence by the choice of ω_4 J is injective. We claim that J is a homeomorphism between $\{0,1\}^{\mathbb{Z}}$ and $J(\{0,1\}^{\mathbb{Z}}) \subset \Lambda_r$. Since $\{0,1\}^{\mathbb{Z}}$ is a metric compact space, and J is injective it is enough to prove sequential continuity for J. Let $s^m \in \{0,1\}^{\mathbb{Z}}$ be a sequence converging in $\{0,1\}^{\mathbb{Z}}$ to s^∞. Then $J(s^m)$ is pre-compact in Λ_r. Let $\bar{J}$ be a limit point of $J(s^m)$. We have:

$$||J(s^\infty) - \sum_i s_i^\infty u_{a_i}^{L'_\epsilon}||_\infty < \omega_4 \tag{4.2}$$

and

$$||J(s^m) - \sum_i s_i^m u_{a_i}^{L'_\epsilon}||_\infty < \omega_4. \tag{4.3}$$

For any compact interval I of $\mathbb{R}$ $||J(s^m) - \bar{J}||_{L^\infty(I)} \to 0$; moreover, since $d(s^m, s^\infty) \to 0$, for any i we get that $s_i^m = s_i^\infty$ provided $m \ge \bar{m}(i)$. Hence passing to the limit in (4.3) we get:

$$||\bar{J} - \sum_i s_i^\infty u_{a_i}^{L'_\epsilon}||_\infty < \omega_4. \tag{4.4}$$

Comparing (4.2) and (4.4), we conclude by theorem 6 that $\bar{J} = J(s^\infty)$. This proves that $J(s^m) \to J(s^\infty)$ and hence that J is sequentially continuous.

Consider now the action of $\mathbb{Z}$ on Λ_r defined by :

$$\text{for } p \in \mathbb{Z} \quad \nu_p(u) = u(\cdot + p).$$

Let $l = [L_\epsilon] + 1 \in \mathbb{N}$. By the same uniqueness argument as that used above it is easy to show that the diagram:

$$\begin{array}{ccc} \Sigma & \xrightarrow{J} & \Lambda_r \\ \sigma\downarrow & & \downarrow\nu_l \\ \Sigma & \xrightarrow{J} & \Lambda_r \end{array}$$

is commutative. The evaluation map:

$$Ev : \Lambda_r \to \mathbb{R}^{2n} \text{ is defined by } \quad Ev(u) = (u(0), \dot{u}(0)).$$

Ev is continuous and by classical continuity results on the Cauchy problem $(Ev)^{-1} : Ev(\Lambda_r) \to \Lambda_r$ is continuous. Moreover the diagram

$$\begin{array}{ccc} \Lambda_r & \xrightarrow{Ev} & \mathbb{R}^{2n} \\ \nu_l \Big\downarrow & & \Big\downarrow \Phi^l \\ \Lambda_r & \xrightarrow{Ev} & \mathbb{R}^{2n} \end{array}$$

is commutative.

Finally define the composition map τ by :

$$\tau : \Sigma = \{0,1\}^{\mathbb{Z}} \to \mathbb{R}^{2n} \quad \text{given by} \quad \tau = Ev \circ J$$

τ is obviously continuous. The last two commutative diagrams implies the thesis that is: $\tau \circ \sigma = \Phi^l \circ \tau$. The proof is complete. □

Remark 12 *Note that (see [4]) transverse intersections between W^u and W^s correspond to non-degenerate critical points of the functional f_ϵ.*

5 Other applications

5.1 Radial systems

The previous arguments can be applied also to study radial systems like:

$$-\ddot{u} + u = |u|^{p-1}u + \epsilon \nabla_u W(t, u) \tag{5.1}$$

with $u \in \mathbb{R}^n$ and $p > 1$. The potential V is $V(u) = \frac{1}{p+1}|u|^{p+1}$.

Let $u_0(t)$ be the unique solution of the scalar problem:

$$-\ddot{u}_0 + u_0 = {u_0}^p, \quad (u_0 > 0), \quad \dot{u}_0(0) = 0, \quad \lim_{|t|\to\infty} u_0(t) = 0. \tag{5.2}$$

Then $Z = \{u_0(\cdot + \theta)\xi \mid \theta \in \mathbb{R} \text{ and } \xi \in S^{n-1}\}$ is a smooth n-dimensional manifold of critical points, diffeomorphic to $S^{n-1} \times \mathbb{R}$.

In [1] it is shown that Z is non-degenerate, *i.e.* $T_{u_{\xi,\theta}} Z = Ker f_0''(u_{\xi,\theta})$ and one readily checks that the arguments developed in the previous sections can be applied also in this situation. $G(u_{\xi,\theta})$ is the usual Melnikov primitive and has the form:

$$G(u_{\xi,\theta}) = \Gamma(\xi, \theta) = -\int_{\mathbb{R}} W(t, \xi u_0(t + \theta))dt. \tag{5.3}$$

Condition 1 becomes:

- There are a constant $\eta > 0$ and a sequence $(U_n)_{n\in\mathbb{Z}}$ of bounded open subsets of $S^{n-1} \times \mathbb{R}$ which satisfy: $\min_{\partial U_n} \Gamma \geq \min_{\bar{U}_n} \Gamma + \eta$ and, π denoting the projection $S^{n-1} \times \mathbb{R} \to \mathbb{R}$, $\pi(U_n) = (c_n, d_n)$ with $\lim_{n\to+\infty} c_n = +\infty$ and $\lim_{n\to-\infty} d_n = -\infty$.

If the above condition is satisfied then the same result as in theorem 4 holds true.

5.2 Forced systems

Here we apply the methods of the last sections in order to find multibump solutions for differential equations of the form:

$$-\ddot{u} + u = \nabla V(u) + \epsilon f(t) \tag{5.4}$$

with $f \in L^\infty$. Assume that (V_1) and (V_2) are satisfied and that V is C^3. For $\epsilon = 0$, 0 is a hyperbolic equilibrium. A simple application of the implicit function theorem shows that in a neighborhood of 0 and for ϵ small enough there exists a unique solution $\gamma_\epsilon(t) \in L^\infty$ of the perturbed system. We want to prove the existence of solutions doubly asymptotic to γ_ϵ.

First we insert the following change of variables:

$$u = x + \gamma_\epsilon \tag{5.5}$$

in equation (5.4) and we obtain for x the following equation:

$$-\ddot{x} + x = \nabla V(x + \gamma_\epsilon) - \nabla V(\gamma_\epsilon). \tag{5.6}$$

If we define:

$$W(\epsilon, t, x) = -V(x) + V(x + \gamma_\epsilon) - \nabla V(\gamma_\epsilon)x - V(\gamma_\epsilon)$$

equation (5.6) becomes:

$$-\ddot{x} + x = \nabla V(x) + \nabla_x W(\epsilon, t, x). \tag{5.7}$$

Since $\gamma_\epsilon = \epsilon\gamma_0 + o(\epsilon)$ we have that $W(\epsilon, t, x) = \epsilon\nabla V(x)\gamma_0(t) + o(\epsilon)W_2(\epsilon, x, t)$. Then (5.7) is in a well-suited form to carry out the arguments of sections 2-3 and 4 (see remark 4).

The Melnikov primitive of system (5.7) is:

$$\Gamma^*(\theta) = -\int_{\mathbb{R}} \nabla V(u_0(t))\gamma_0(t - \theta)dt.$$

Since u_0 is a solution of $-\ddot{x} + x = \nabla V(x)$, one finds that:

$$\Gamma^*(\theta) = -\int_{\mathbb{R}} (-\ddot{u}_0 + u_0)\gamma_0(t - \theta)dt. \tag{5.8}$$

Moreover since γ_ϵ solves (5.4) γ_0 solves the equation $-\ddot{\gamma}_0 + \gamma_0 = f(t)$. Hence integrating by parts in (5.8) we have that:

$$\Gamma^*(\theta) = -\int_R f(t - \theta)u_0(t)dt = \Gamma(\theta).$$

Then we can apply to equation (5.4) all the results of the last sections proving the existence of infinitely many homoclinics and of solutions with infinitely many bumps provided the Melnikov primitive Γ satisfies conditions 1 or 2.

5.3 Partial differential equations of Schrödinger type

Thanks to the generality of our approach we can handle partial differential equations such as:

$$-\Delta u + u = |u|^{p-2}u + \epsilon\nabla_u W(x, u) = 0, \quad |u(x)| \to 0 \text{ as } |x| \to \infty \tag{5.9}$$

where $u : \mathbb{R}^n \to \mathbb{R}$. We assume that $2 < p < 2^* = 2n/(n-2)$ and that:

- (W_3) $W \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$, $W(x, 0) = 0$, $\nabla_u W(x, 0) = 0$, D^2W is continuous uniformly with respect to $x \in \mathbb{R}^n$; moreover $|D_u^2 W(x, \cdot)| \leq C_1|u|^{q-2} + C_2$ for some q with $2 < q < 2^*$.

In this paragraph we will work in the Sobolev space $E = W^{1,2}(\mathbb{R}^n, \mathbb{R})$.

It is well known that the unperturbed equation has a unique positive solution z_0 such that $\nabla z_0(0) = 0$ and that z_0 has an exponential decay as $|x| \to \infty$.

The perturbed functional is $f_\epsilon = (\frac{1}{2})||u||^2 + F(u) + \epsilon G(u)$ where:

$$F(u) = -\frac{1}{p}\int_{\mathbb{R}^n} |u|^p dx \quad \text{and} \quad G(u) = -\int_{\mathbb{R}^n} W(x,u).$$

The manifold $Z = \{z_0(\cdot + \theta) \mid \theta \in \mathbb{R}^n\}$ is a smooth n-dimensional non-degenerate critical manifold for f_0 (see [2] and references therein for a proof).

Still in this situation, using the Sobolev embedding theorems, it is possible to apply the arguments of sections 2 and 3.

The Melnikov primitive $\Gamma : \mathbb{R}^n \to \mathbb{R}$ is defined by:

$$\Gamma(\theta) = -\int_{\mathbb{R}^n} W(x, z_0(x+\theta))dx.$$

Assume that Γ satisfies:

Condition 4 *There are $\eta > 0$ and a sequence $(U_n)_{n\in\mathbb{Z}}$ of bounded open subsets of $\mathbb{R}^n$ which satisfy:*

(i) $\Gamma_{|U_n}$ *attains its minimum at some $a_n \in U_n$ and $\Gamma_{|\partial U_n} \geq \eta + \Gamma(a_n)$;*

(ii) $\min_{x\in U_n} |x| \to \infty$ *as* $|n| \to \infty$.

By the same arguments as in section 2, condition 4 allows to prove the existence of infinitely many k-bump solutions of (5.9).

Moreover, we can prove the existence of solutions with infinitely many bumps: let us define a norm which is analogous to the norm $|\cdot|_\theta$ defined in section 3.

Fix $L > 8$ and $(\theta_1, \ldots, \theta_k) \in \mathbb{R}^{nk}$ such that $\min_i |\theta_{i+1} - \theta_i| > L$. For $x \in \mathbb{R}^n$ we define:

$$R(x) = \sup\{R \mid B(x,R) \text{ contains at most one } \theta_i\}$$

where $B(x,R)$ is the ball of center x and radius R. Note that $R(x) \geq L/2$. Next we define the norm on E

$$||u||_\theta = \max\Big\{|u|_\theta, ||u||_{W^{1,2}(D)}\Big\},$$

where $|u|_\theta = \sup_{x\in\mathbb{R}^n} ||u||_{W^{1,2}(B(x,R(x)/4))}$ and $D = \Big\{x \in \mathbb{R}^n \;\Big|\; \min_i |x - \theta_i| \geq 2M \;\text{ with } M = \text{diam}\{\theta_1, \ldots, \theta_k\}\Big\}$.

Using the above norm, with arguments similar to those of the previous sections, it is possible to prove that:

Theorem 8 *Let (W_3) and condition 4 hold. $\forall\omega > 0$, $\exists\epsilon_4 > 0$ such that $\forall\epsilon \in (-\epsilon_4, 0) \cup (0, \epsilon_4)$ there exists L_ϵ such that for any sequence $(i_l)_{l\in K\subset\mathbb{Z}}$ with $\inf_l |dist(U_{i_{l+1}}, U_{i_l})| > L_\epsilon$ there are $\theta_l \in U_{i_l}$ and a solution u_ϵ of (5.9) which satisfies:*

$$||u_\epsilon - \sum_l u_{\theta_l}||_\infty \leq \omega.$$

6 Appendix

In this appendix we prove lemmas 8, 9, 10, 17.

Proof of lemma 8-(i):

By the definition of f_ϵ we have that for $|\epsilon| \le 1$: $|f_\epsilon(s+r) - f_\epsilon(s) - f_\epsilon(r)| =$

$$|\frac{1}{2}||s+r||^2 - \frac{1}{2}||s||^2 - \frac{1}{2}||r||^2 - \int_{\rm I\!R} (V(s+r) - V(s) - V(r)) - \epsilon \int_{\rm I\!R} (W(t,s+r) - W(t,s) - W(t,r))| \le$$

$$|(s,r)| + \int_{\rm I\!R} |V(s+r) - V(s) - V(r)| + \int_{\rm I\!R} |W(t,s+r) - W(t,s) - W(t,r)|. \tag{6.1}$$

Obviously

$$|(s,r)| \le \int_{\rm I\!R} |r||s| + |\dot{r}||\dot{s}|. \tag{6.2}$$

In order to estimate the other terms in (6.1) consider the following function of the variable s, $V_r(s)$ $= V(s+r) - V(s) - V(r)$. We have that :

$$|V_r(s)| = |V_r(s) - V_r(0)| \le \sup_{\lambda \in [0,1]} |DV_r(\lambda s)||s| = \sup_{\lambda \in [0,1]} |DV(\lambda s + r) - DV(\lambda s)||s|.$$

Moreover:

$$|DV(\lambda s + r) - DV(\lambda s)| \le \sup_{\mu \in [0,1]} |D^2 V(\mu(\lambda s + r) + (1-\mu)\lambda s)||r(t)|.$$

Hence the last 2 formulas yield:

$$|V(s+r) - V(s) - V(r)| \le \sup_{\mu,\lambda \in [0,1]} |D^2 V(\mu r + \lambda s)||r||s|.$$

Since $D^2 V$ is continuous and $||\mu r + \lambda s||_\infty \le 2b$ there exists a positive constant $C'(b)$ such that:

$$|D^2 V(\mu r + \lambda s)||r||s| \le C'(b)|r||s|$$

and this implies that

$$\int_{\rm I\!R} |V(s+r) - V(s) - V(r)| \le C'(b) \int_{\rm I\!R} |r||s|. \tag{6.3}$$

Using that $D^2 W(t,u)$ is bounded on bounded subsets of ${\rm I\!R}^n$ uniformly with respect to t we can obtain:

$$\int_{\rm I\!R} |W(t,s+r) - W(t,s) - W(t,r)| \le C''(b) \int_{\rm I\!R} |r||s|. \tag{6.4}$$

Lemma 8-(i) is a consequence of (6.1), (6.2), (6.3) and (6.4).

Proof of lemma 8-(ii):

He have: $|(f'_\epsilon(s+r) - f'_\epsilon(r), v)| \le$

$$|(s+r,v) - (r,v)| + \int_{\rm I\!R} |DV(s+r) - DV(r)||v| dt + \int_{\rm I\!R} |DW(t,s+r) - DW(t,r)| dt$$

$$\le \int_{\rm I\!R} |s||v| + |\dot{s}||\dot{v}| + \int_{\rm I\!R} \sup_{\lambda \in [0,1]} |D^2 V(r + \lambda s)||v||s| dt + \int_{\rm I\!R} \sup_{\lambda \in [0,1]} |D^2 W(t, r + \lambda s)||v||s| dt$$

since $||r+\lambda s||_\infty \leq 2b$ and $|D^2W(t,\cdot)|$ is bounded on bounded subsets of $\mathbb{R}^n$ uniformly in t we have that:

$$\begin{aligned} &\leq \int_{\mathbb{R}} |s||v| + |\dot{s}||\dot{v}| + \int_{\mathbb{R}} C'(b)|s||v|dt + \int_{\mathbb{R}} C'(b)|v||s|dt \\ &\leq C(b)\int_{\mathbb{R}} |s||v| + |\dot{s}||\dot{v}|.\square \end{aligned}$$

Proof of lemma 9-(i):

By the property (P_θ) of lemma 7 there is $Y \in E$ such that $||Y|| = 1$, supp $Y \subset [m_{i,i+1}-2, m_{i-1,i}+2]$, where $m_{i,i+1} = -(\theta_i+\theta_{i+1})/2$ and $|f_0'(u^L_{\theta_1}+\cdots+u^L_{\theta_k}+v)|_\theta \leq 5(f_0'(u^L_{\theta_1}+\cdots+u^L_{\theta_k}+v), Y)$.

It is clear that for $j \neq i$, Y and $u^L_{\theta_j}$ have disjoint supports. Since by (V_1), $|\nabla V(x)|/|x|$ is bounded on bounded subsets of $\mathbb{R}^n$, we have that:

$$|(f_0'(u^L_{\theta_1}+\cdots+u^L_{\theta_k}+v),Y)| = |(u^L_{\theta_i}+v,Y) - \int_{\mathbb{R}} \nabla V(u^L_{\theta_i}+v)Y| \leq (||u^L_0||+1) + C\int_{\mathbb{R}} |u^L_{\theta_i}+v||Y|.$$

This clearly implies $|(f_0'(u^L_{\theta_1}+\ldots+u^L_{\theta_k}+v),Y)| \leq (1+C)(||u^L_0||+1)$ and we get (i).

Using (W_1) the estimate **9-(ii)** is obtained in the same way. **9-(iii)** and **9-(iv)** can also be easily proved, using (P_θ) and the fact that $|D^2V|$ and $\sup_t |D^2W(t,\cdot)|$ are bounded on bounded subsets of $\mathbb{R}^n$.

Proof of lemma 9-(v) Let $Y \in E$ satisfy

$$|(f_0''(\sum_{i=1}^k u^L_{\theta_i}+v) - f_0''(\sum_{i=1}^k u^L_{\theta_i}))X|_\theta \leq 5((f_0''(\sum_{i=1}^k u^L_{\theta_i}+v) - f_0''(\sum_{i=1}^k u^L_{\theta_i}))X, Y)$$

with $||Y|| \leq 1$ and supp $Y \subset [m_{j,j+1}-2, m_{j-1,j}+2]$ for some j. Then

$$((f_0''(\sum_{i=1}^k u^L_{\theta_i}+v) - f_0''(\sum_{i=1}^k u^L_{\theta_i}))X,Y) = \int_{m_{j,j+1}-2}^{m_{j-1,j}+2} (D^2V(u^L_{\theta_j}+v) - D^2V(u^L_{\theta_j}))XY.$$

Now, since D^2V is uniformly continuous on bounded subsets of $\mathbb{R}^n$, using the fact that $||v||_\infty \leq 2|v|_\theta$ we can write

$$||D^2V(u_{\theta_j}+v) - D^2V(u_{\theta_j})||_\infty \leq r(|v|_\theta) \tag{6.5}$$

with $\lim_{s\to 0} r(s) = 0$. Moreover

$$\int_{m_{j,j+1}-2}^{m_{j-1,j}+2} |X||Y| \leq ||Y|| \cdot ||X||_{W^{1,2}(m_{j,j+1}-2,m_{j-1,j}+2)} \leq 3|X|_\theta. \tag{6.6}$$

Combining (**??**) and (6.6) we get the desired estimate.

Proof of lemma 10-(i)

Since 0 is a hyperbolic equilibrium point of (2.1), u_0 and $\dot{u}_0$ have exponential decay, i.e. there are two positive constants C and C' such that

$$|u_0(t)|, |\dot{u}_0(t)| \leq C\exp(-C'|t|). \tag{6.7}$$

Since u_0 solves equation 2.1 , (6.7) implies by (V_1) that $\ddot{u}_0$ and $\dddot{u}_0$ have also exponential decay. So we can assume that

$$|\ddot{u}_0(t)|, |\dddot{u}_0(t)| \leq C\exp(-C'|t|).$$

By the definition of u_0^L it is easy to see that these estimates imply the existence of constants C'' and C''' such that for all $L > 8$

$$||u_0 - u_0^L|| + ||\dot{u}_0 - \dot{u}_0^L|| + ||\ddot{u}_0 - \ddot{u}_0^L|| \leq C''\exp(-C'''L)$$

(Remember that u_0 and u_0^L coincide on $(-L/4, L/4)$).

Proof of lemma 10-(ii)

Let $Y \in E, ||Y|| \leq 1$ have its support in $I_i + [-2,2]$ for some $i \in \{1,\ldots,k\}$, where $I_i = [-(\theta_i+\theta_{i+1})/2, -(\theta_i+\theta_{i-1})/2]$. Then for $j \neq i$ supp $u_{\theta_j}^L \cap$ supp $Y = \emptyset$ and since u_{θ_i} solves (2.1):

$$\begin{aligned}(f_0'(u_{\theta_1}^L + \cdots + u_{\theta_k}^L), Y) &= (u_{\theta_i}^L, Y) - \int_{\mathbb{R}} \nabla V(u_{\theta_i}^L)Y \\ &= (u_{\theta_i}^L - u_{\theta_i}, Y) - \int_{\mathbb{R}} (\nabla V(u_{\theta_i}^L) - \nabla V(u_{\theta_i}))Y.\end{aligned}$$

Hence, by 10-(i) and since D^2V is bounded on bounded subsets of $\mathbb{R}^n$ there is a constant C such that:

$$|(f_0'(u_{\theta_1}^L + \ldots + u_{\theta_k}^L), Y)| \leq C\exp(-C'L).$$

Therefore 10-(ii) holds by lemma 7.

Proof of lemma 10-(iii) Let Y satisfy $||Y|| \leq 1$, supp $Y \subset I_i + [-2,2]$.

We have:

$$\begin{aligned}|(f_0''(u_{\theta_1}^L + \ldots + u_{\theta_k}^L + v)(\lambda_1\dot{u}_{\theta_1}^L + \ldots + \lambda_k\dot{u}_{\theta_k}^L), Y)| &= |(f_0''(u_{\theta_i}^L + v)\lambda_i\dot{u}_{\theta_i}^L, Y)| \\ &\leq |\lambda_i| 3 |f_0''(u_{\theta_i}^L + v)\dot{u}_{\theta_i}^L|_\theta ||Y|| \\ &\leq 3C_4|\lambda_i||\dot{u}_{\theta_i}^L|_\theta\end{aligned}$$

by lemma 9-(iii). So we get the first estimate from property (P_θ) of lemma 7.

Now assume that D^2V is locally Lipschitz continuous. Then

$$|(f_0''(u_{\theta_i}^L + v)\lambda_i\dot{u}_{\theta_i}^L, Y)| = |\lambda_i||(\dot{u}_{\theta_i}^L, Y) - \int_{\mathbb{R}} D^2V(u_{\theta_i}^L + v)\dot{u}_{\theta_i}^L Y|.$$

Since

$$(\dot{u}_{\theta_i}, Y) - \int_{\mathbb{R}} D^2V(u_{\theta_i})\dot{u}_{\theta_i}Y = 0$$

we get (using that supp $Y \subset I_i \cup I_{i-1} \cup I_{i+1}$ and $||Y|| \leq 1$)

$$\begin{aligned}|(f_0''(u_{\theta_i}^L + v)\lambda_i\dot{u}_{\theta_i}^L, Y)| &\leq |\lambda_i|(|(\dot{u}_{\theta_i}^L - \dot{u}_{\theta_i}, Y)| + \int_{\mathbb{R}} |D^2V(u_{\theta_i}^L + v) - D^2V(u_{\theta_i})||\dot{u}_{\theta_i}^L||Y| \\ &+ \int_{\mathbb{R}} |D^2V(u_{\theta_i})||\dot{u}_{\theta_i}^L - \dot{u}_{\theta_i}||Y|) \\ &\leq |\lambda_i|(||\dot{u}_{\theta_i}^L - \dot{u}_{\theta_i}|| + C(|v|_\theta + ||u_{\theta_i}^L - u_{\theta_i}||)||\dot{u}_{\theta_i}^L|| + C||\dot{u}_{\theta_i}^L - \dot{u}_{\theta_i}||)\end{aligned}$$

The second estimate is then a consequence of 10-(i) and of property (P_θ).

Proof of lemma 17 :

We are going to prove that:

$$||\tilde{w}_\epsilon||_{W^{1,2}(J)} = O(\exp(-C(\theta_2-\theta_1)) \tag{6.8}$$

where $J = (-(\theta_1+\theta_2)/2 - 2, -(\theta_1+\theta_2)/2 + 2)$.
For $k \le (\theta_2-\theta_1)/2 - L/4$ let $G_k = (-\theta_2 + L/4 + k, -\theta_1 - L/4 - k)$.

We already know by lemmas 13 and 14 that $||\tilde{w}_\epsilon||_{W^{1,2}(G_0)} \le \epsilon + C\exp(-CL)$. Now, since $G_0 \cap \text{supp } u^L_{\theta_i} = \emptyset$ for all i, by the definition of w and $\bar{w}_\epsilon$, we have

$$-\ddot{\tilde{w}}_\epsilon + \tilde{w}_\epsilon = \nabla V(\tilde{w}_\epsilon) + \nabla W(t,\tilde{w}_\epsilon) \quad \text{on } G_0 \tag{6.9}$$

For $k \ge 1$ let φ_k be the function defined on G_0 by: $\varphi_k = 1$ on G_k; $\varphi_k = 0$ outside G_{k-1}; φ_k is continuous on G_0 and it is linear on each component of $G_{k-1}\backslash G_k$. By (6.9)

$$\int_{G_0} (\dot{\tilde{w}}_\epsilon, \varphi_k \dot{\tilde{w}}_\epsilon + \dot{\varphi}_k \tilde{w}_\epsilon) + (\tilde{w}_\epsilon, \varphi_k \tilde{w}_\epsilon) - (\nabla V(\tilde{w}_\epsilon) + \epsilon \nabla W(t,\tilde{w}_\epsilon), \varphi_k \tilde{w}_\epsilon) = 0$$

This implies (since $|\dot{\varphi}_k| \le 1$)

$$\int_{G_k} |\dot{\tilde{w}}_\epsilon|^2 + |\tilde{w}_\epsilon|^2 \le \int_{G_k} |\nabla V(\tilde{w}_\epsilon) + \epsilon\nabla W(t,\tilde{w}_\epsilon)||\tilde{w}_\epsilon| + \int_{G_{k-1}\backslash G_k} |\dot{\tilde{w}}_\epsilon||\tilde{w}_\epsilon| + |\nabla V(\tilde{w}_\epsilon) + \epsilon\nabla W(t,\tilde{w}_\epsilon)||\tilde{w}_\epsilon|. \tag{6.10}$$

Now, by (V_1), (W_1), $\lim_{x\to 0} |\nabla V(x)|/|x| = 0$ and there is a constant C such that $|\nabla W(t,x)| \le C|x|$ for $|x| \le 1$. Hence, by lemmas 13 and 14, for L large enough and ϵ small enough,

$$\int_{G_k} |\nabla V(\tilde{w}_\epsilon) + \epsilon \nabla W(t,\tilde{w}_\epsilon)||\tilde{w}_\epsilon| \le \frac{1}{2}\int_{G_k} |\tilde{w}_\epsilon|^2. \tag{6.11}$$

(6.10) then implies

$$\begin{aligned}\int_{G_k} |\dot{\tilde{w}}_\epsilon|^2 + |\tilde{w}_\epsilon|^2 &\le 2\int_{G_{k-1}\backslash G_k} |\dot{\tilde{w}}_\epsilon||\tilde{w}_\epsilon| + |\nabla V(\tilde{w}_\epsilon) + \epsilon\nabla W(t,\tilde{w}_\epsilon)||\tilde{w}_\epsilon| \\ &\le C''\int_{G_{k-1}\backslash G_k} |\dot{\tilde{w}}_\epsilon|^2 + |\tilde{w}_\epsilon|^2\end{aligned}$$

where C'' is some constant independent of L, ϵ, k. Setting $R_k = ||\tilde{w}_\epsilon||^2_{W^{1,2}(G_k)}$ we get $R_k \le C''(R_{k-1} - R_k)$. Hence

$$R_k \le \frac{C''}{C''+1} R_{k-1}.$$

Therefore

$$R_k \le (\frac{C''}{C''+1})^k (\epsilon^2 + \exp(-CL)). \tag{6.12}$$

If we take $k = [(\theta_2-\theta_1)/2 - L/4] - 2$ then $J \subset G_k$ and (6.12) implies the estimate of lemma 17.

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